Lectures on Fixed Point Theory
Mini-Course
XV Encontro Brasileiro de Topologia
Rio Claro, SP Brazil
July 10 - 14, 2006

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Contents

Preface

Topological fixed point theory plays an important role in the development of algebraic topology during the early decades of the twentieth century. These notes are intended as a brief introduction to the subject with a focus on the Lefschetz fixed point theorem and its converse. While there is overlap in content with the similar mini-course I gave at the XIV EBT in Campinas in 2004, the present notes complement those by including new topics such as combinatorial fixed point theorems, and some recent development on the finiteness of the Reidemeister number and its connection with (geometric) group theory.

Throughout some proofs are omitted or are only sketched. The basic references are [**1**], [**6**], [**11**], [**13**], [**15**], and [**16**]. The survey article [**2**] gives a very nice historical account of the early development of the subject. Finally, [**4**] and [**9**] present some of the most recent developments of the subject.

I thank the scientific and organizing committees for the kind invitation and the opportunity for me to give this course again.

Lecture I - Combinatorial Fixed Point Theorems

We first recall some of the earliest results in topological fixed point theory, including classical theorems of Bolzano, Brouwer, and Borsuk. Combinatorial analogs due to Sperner and to Tucker are presented. Recent developments in combinatorial fixed point theory will be discussed.

1. The three B's in fixed point theory

Like Bach, Beethoven, and Brahms, who are also known as the three B's in classical music, we have Bolzano, Brouwer, and Borsuk (among other B's) in topological fixed point theory. These are the early pioneers in the field. We begin with the first topological fixed point theorem, also known as the *Intermediate Value Theorem*, which was first proved by Bernard Bolzano (1781 - 1848) in 1817.

THEOREM 1.1. *Let* $f : [-1, 1] \to \mathbb{R}$ *be a continuous function such that* $f(-1) < 0$ *and* $f(1) > 0$*. Then there exists a point* $c \in [-1, 1]$ *such that* $f(c) = 0$*.*

This is equivalent to the following Brouwer Fixed Point Theorem in dimension 1.

THEOREM 1.2. *Let* D^n *denote the closed* n*-disk in* $\mathbb{R}^n$ *and* $f : D^n \to D^n$ *a (continuous) map. Then there exists a point* $x_0 \in D^n$ *such that* $f(x_0) = x_0$*.*

The Intermediate Value Theorem (IVT) is a topological result about the real line. One can obtain the one dimensional Brouwer fixed point theorem from the IVT.

THEOREM 1.3. *Let* $f : [-1,1] \to [-1,1]$ *be a continuous function. There exists* $c \in [-1,1]$ *such that* $f(c) = c$.

PROOF. Let $h(x) = x - f(x)$. The map h is continuous on $[-1,1]$. If $f(1) = 1$ or $f(-1) = -1$ then we are done. Now, suppose $f(1) \neq 1$ and $f(-1) \neq -1$. Then, $h(-1) < 0$ and $h(1) > 0$. By the IVT, there exists $c \in [-1,1]$ such that $h(c) = 0$, i.e., $f(c) = c$. ☐

REMARK 1.1. The Brouwer Fixed Point Theorem was first published in 1909 by Luitzen Egbertus Jan Brouwer (1881 - 1966) for continuous maps on D^3, then in 1910 for differentiable maps on D^n. Finally, he proved the general version in 1912. Interestingly, Brouwer's theorem was predated by the following result of Piers Bohl (1865 - 1921) in 1904. Bohl's proof requires differentiability of the map.

THEOREM 1.4. *There is no (differentiable) map* $f : D^n \to \mathbb{R}^n - \{0\}$ *such that* f *is the identity on the boundary* ∂D^n.

This is essentially the same as the following *no-retraction* theorem.

THEOREM 1.5. *There is no continuous map* $r : D^n \to \partial D^n$ *with* $r(x) = x$ *for all* $x \in \partial D^n$ *where* $\partial D^n = S^{n-1}$ *is the boundary of* D^n.

REMARK 1.2. In 1931, K. Borsuk observed that Theorem 1.2 is equivalent to Theorem 1.5.

This leads us to another beautiful classical fixed point theorem, known as the Borsuk-Ulam Theorem, first conjectured by S. Ulam and then proved by Karol Borsuk (1905 - 1982) in 1933.

THEOREM 1.6. *Let* $f : S^2 \to \mathbb{R}^2$ *be a map. Then exists a point* $z \in S^2$ *such that* $f(z) = f(-z)$ *where* $-z$ *denotes the antipodal point of* z.

If we write $f(z) = (h_1(z), h_2(z))$ then we can interpret Theorem 1.6 as follows:

At any given time, there is a location z *on earth* (S^2) *whose temperature* ($h_1(z)$) *and barometric pressure* ($h_2(z)$) *are the same as those at its antipodal point* $-z$.

In fact, the Borsuk-Ulam theorem holds for all dimension. Another interpretation of the theorem in dimension 3 is the following so-called *Ham Sandwich Theorem.*

THEOREM 1.7. *Given a slice of bread, a slice of ham, and a slice of cheese, there is a (single) cut that will bisect each slice simultaneously.*

Theorem 1.6 is equivalent to another theorem proven by Lusternik and Schnirelmann in 1930 based upon their work on topological methods in analysis.

THEOREM 1.8. *If* S^n *is covered by* $n+1$ *(non-empty) closed subsets* $C_1, ..., C_{n+1}$*, then for some* i*,* $1 \leq i \leq n+1$*,* C_i *contains a pair of antipodes.*

REMARK 1.3. There is a VAST literature about Borsuk-Ulam theorems and related topics. For example, type "Borsuk Ulam" in *MathSciNet*, you will get more than 330 articles. The bibliography of an survey article by H. Steinlein (1985) contains no less than 400 items (and growing). I suspect that there are at least 5-10 papers published every year about Borsuk-Ulam or related topics.

2. Sperner's Lemma

All of the theorems mentioned in the previous section are of the type of existence results. None of the (original) proofs of these results indicate *how* one would proceed to find fixed points or antipodal points. We consider elementary *combinatorial* versions of these theorems.

Consider a triangle Δ and label the three vertices by $\{1, 2, 3\}$.

Given *any* triangulation $\mathcal{T}$ of Δ, we use a *Sperner labelling* of $\mathcal{T}$ so that $v_1, v_2 \in \{1, 2\}, v_3, v_4 \in \{1, 3\}, v_5, v_6 \in \{2, 3\}$. More generally, a Sperner labelling is defined as follows.

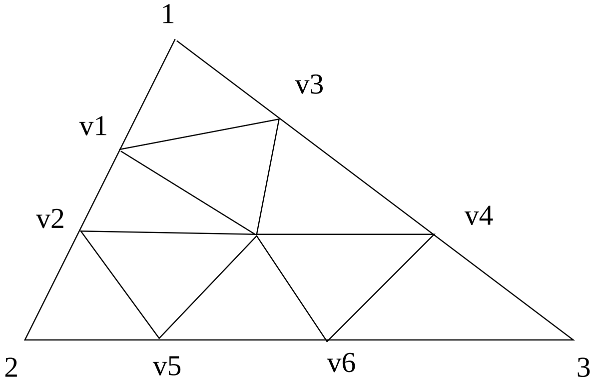

FIGURE 1

Let Δ^d be the d-dimensional simplex and $\mathcal{T}$ be a triangulation of Δ^d. Given a vertex v of $\mathcal{T}$, the *carrier* $carr(v)$ of v is the lowest dimensional face of Δ^d that contains v.

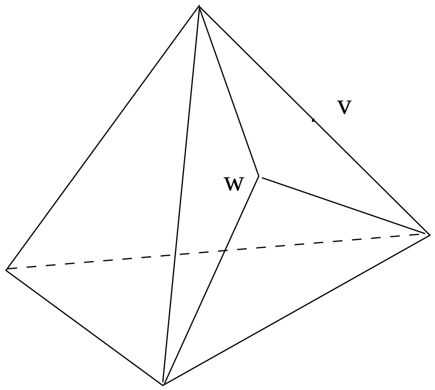

FIGURE 2

A *Sperner labelling* of $\mathcal{T}$ is a labelling (i.e., a function $\ell : V(\mathcal{T}) \to \{1, 2, ..., d+1\}$) so that $\ell(v) \in \{\ell(v_i) | v_i \in carr(v)\}$. A d-simplex σ of $\mathcal{T}$ is *complete* if $\{\ell(v) | v \text{ is a vertex of } \sigma\} = \{1, 2, ..., d+1\}$. In 1928, Emmanuel Sperner (1905 - 1980) proved the following result now known as the Sperner's Lemma.

THEOREM 2.1. *For any Sperner labelling of any triangulation of* Δ^d*, the number of complete d-simplices is odd. In particular, there exists at least one complete d-simplex.*

In fact, Sperner's Lemma is equivalent to the Brouwer Fixed Point Theorem. There are many proofs of the Sperner's Lemma, some non-constructive and some constructive. Here we sketch a non-constructive proof for the case $d = 2$.

PROOF. Given a triangulation $\mathcal{T}$ of Δ^2, a 1-face of a 2-simplex in $\mathcal{T}$ is a *door* if it contains the labels 1 and 2. Now suppose we have a Sperner labelling:

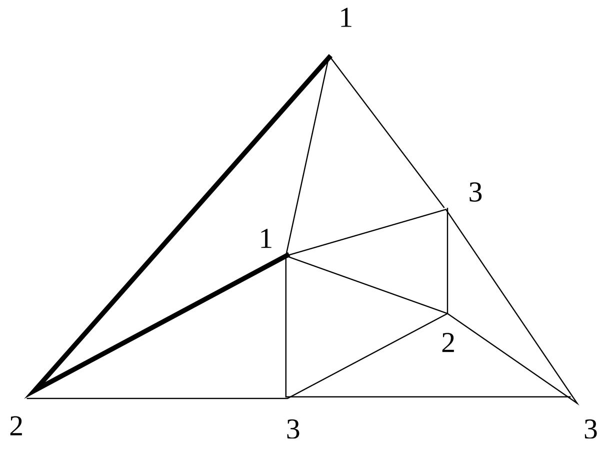

FIGURE 3

Lay down a cashew nut next to each door, one on each side of an interior door and just one on the inside of a boundary door. Clearly, the number of cashew nuts is congruent to the number of cashew nuts next to the boundary doors, modulo 2. It is easy to see that the number of cashew nuts next to the boundary doors is congruent to 1 modulo 2 (this is the Sperner's Lemma in dimension 1). On the other hand, we have

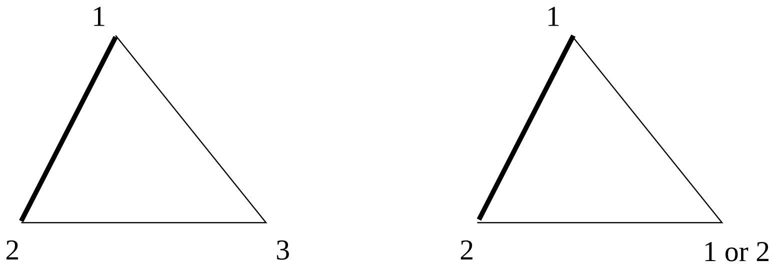

FIGURE 4

so that the number of cashew nuts inside a (room) 2-simplex is odd if and only if the simplex is complete. This shows that there is an odd number of complete 2-simplices. ☐

Another proof (constructive) simply starts with a boundary door and follow the doors. This leads us a way to find a complete 2-simplex.

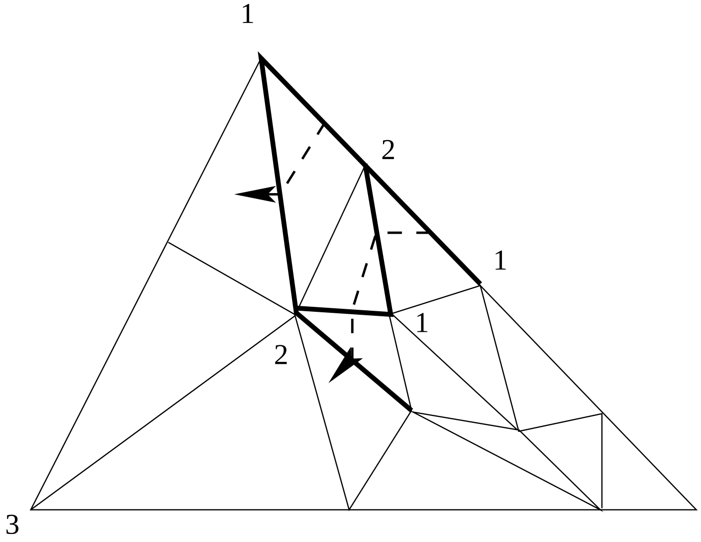

FIGURE 5

Since Sperner's Lemma is known to be equivalent to the Brouwer Fixed Point Theorem, a constructive proof of Sperner's Lemma gives rise to a constructive proof of the Brouwer Fixed Point Theorem.

3. Tucker's Lemma

In 1945, Albert Tucker (1905 - 1995) gave a combinatorial analog of the Borsuk-Ulam Theorem, now known as the Tucker's Lemma.

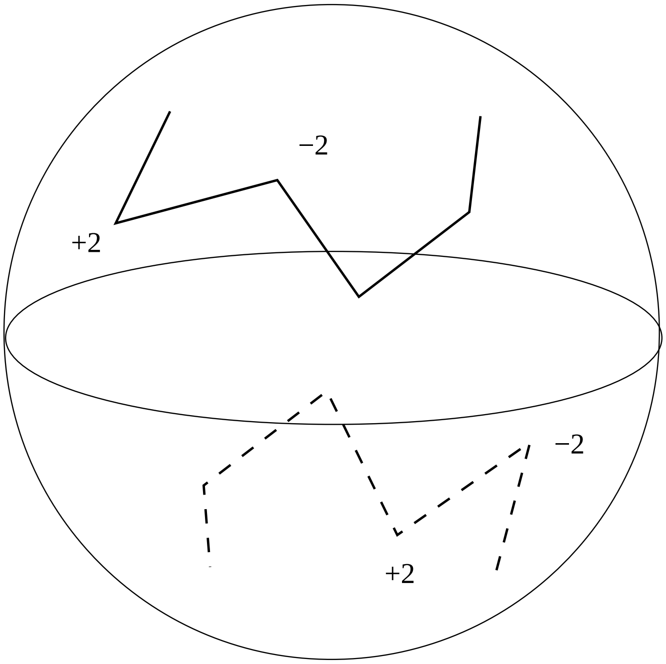

FIGURE 6

Let $\mathcal{T}$ be a triangulation of the unit d-sphere S^d. We say that $\mathcal{T}$ is *symmetric* if whenever σ is a simplex of $\mathcal{T}$, so is its antipode $-\sigma$. A labelling ℓ of S^d by $\{\pm 1, ..., \pm d\}$ is a *Tucker labelling* if $\ell(-v) = -\ell(v)$ for all vertices v of a symmetric triangulation.

THEOREM 3.1. *For any Tucker labelling ℓ of a symmetric triangulation of S^d, there exists an edge $e = \langle v_0, v_1 \rangle$ such that $\ell(v_0) + \ell(v_1) = 0$.*

Like Sperner's Lemma, which is equivalent to the Brouwer Fixed Point Theorem, Tucker's Lemma is equivalent to the Borsuk-Ulam Theorem. Again, a constructive proof of Tucker's Lemma gives rise to a constructive proof of the Borsuk-Ulam Theorem. To see how Tucker's lemma may be used to prove the Borsuk-Ulam Theorem, let us start with a continuous map $f : S^d \to \mathbb{R}^d$. Write $g(z) = f(z) - f(-z)$. Then the Borsuk-Ulam Theorem is equivalent to saying that $g(z) = 0$ must have a solution in S^d. Suppose that there is no solution. Then let $h(z) := \frac{g(z)}{|g(z)|}$ and so $h : S^d \to S^{d-1} \subset \mathbb{R}^d$. Triangulate S^d symmetrically and label each vertex v by the label of the axis $\{\pm 1, \pm 2, ..., \pm d\}$ closest to $h(v)$. This turns out to be a Tucker labelling of S^d. Thus, it must have an edge whose label sum is zero. This contradicts the uniform continuity of h and so $g(z) = 0$ for some $z \in S^d$.

REMARK 3.1. One equivalent form of the Borsuk-Ulam Theorem is by means of group action. A space X is a G-space where G is a group means that G acts on X continuously, i.e., there is a continuous map $\Phi : G \times X \to X$ such that $\Phi(g_1, \Phi(g_2, x)) = \Phi(g_1 g_2, x)$. Take $G = \mathbb{Z}_2 = \{\pm 1\}$ and let G act on S^d via the antipodal action, i.e., $\Phi(1, z) = z, \Phi(-1, z) = -z$. The G-action on $\mathbb{R}^d$ is also the antipodal action symmetric with respect to the origin. For every $f : S^d \to \mathbb{R}^d$, the map $g(z) = f(z) - f(-z)$ is G-equivariant, i.e., $g(-z) = -g(z)$. Then the Borsuk-Ulam is equivalent to the following.

THEOREM 3.2. *For any G-map $\varphi : S^d \to \mathbb{R}^d$, $\varphi^{-1}(0) \neq \emptyset$.*

A natural question to ask is: *How big is $\varphi^{-1}(0)$?*

4. Theorems of Dyson and of Yang

In a short paper published in the *Annals of Mathematics* in 1951, Freeman J. Dyson proved the following beautiful theorem.

THEOREM 4.1. *Let $f : S^2 \to \mathbb{R}$ be a continuous map. Then there exist two orthogonal diameters whose endpoints are mapped to the same value under f.*

While Dyson's theorem does not include the classical Borsuk-Ulam theorem as a special case, it gives more information about the zero set of $g(z) = f(z) - f(-z)$. A far reaching generalization of Dyson's result was given by Chung-Tao Yang (1923 - 2005) in 1954 and 1955. As a special case, Yang proved the following result.

THEOREM 4.2. *For every continuous map* $f : S^{nd} \to \mathbb{R}^d$ *there exist* n *mutually orthogonal diameters whose* $2n$ *endpoints are mapped to the same value under* f.

In fact, Yang proved a lot more. The following result, known as the Bourgin-Yang Theorem, was first obtained by D. Bourgin in the early 1950s.

THEOREM 4.3. *Given a continuous map* $f : S^{nd} \to \mathbb{R}^d$, *the set* $A_f = \{z|f(z) = f(-z)\}$ *has "cohomological size" at least* $nd - d$.

REMARK 4.1. This result has been generalized and sophisticated cohomological indices have been defined and used in analysis.

In 2004, at the Park City Mathematics Institute, Francis Su asked: What are the combinatorial analogs of Yang's Theorem 4.2? Are there combinatorial and constructive proofs of these results?

Similar to Tucker's Lemma, there is a combinatorial analog (with a combinatorial proof) which is equivalent to Dyson's Theorem 4.1 ($d = 1, n = 2$) [P. Jayawant and P.W., preprint 2006]. For other values - unknown.

Lecture II - The Lefschetz Fixed Point Theorem

We introduce the classucal fixed point index and the Lefschetz number. The celebrated Lefschetz Fixed point Theorem is discussed and we investigate the validity of the converse via Nielsen-Wecken theory. Jiang's counter-example to the Nielsen conjecture is presented.

5. The Lefschetz Number

In dimension 1, the Intermediate Value Theorem says that the graph of f must cross the diagonal.

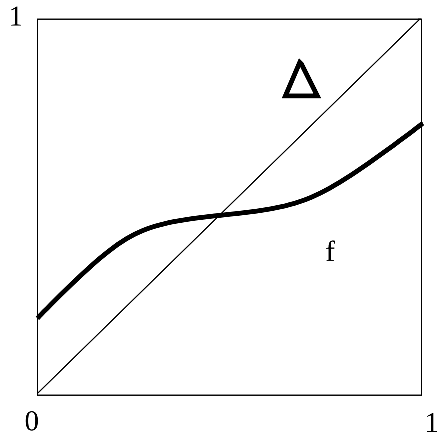

Figure 7

Lefschetz took the same idea, namely, regarding fixed points as intersections between the graph of $f : M \to M$ and the diagonal $\Delta_M = \{(x, x) | x \in M\}$. On the other hand, for a triangulated space M, if a simplex σ is mapped into itself under f then the Brouwer Fixed Point Theorem implies that f must have a fixed point in σ.

For simplicity, let M be a compact connected oriented smooth n-manifold (without boundary) and $f : M \to M$ a smooth map. We deform (or perturb) f to a map f' so that the graph of f' has only regular (or transverse) intersections with Δ_M.

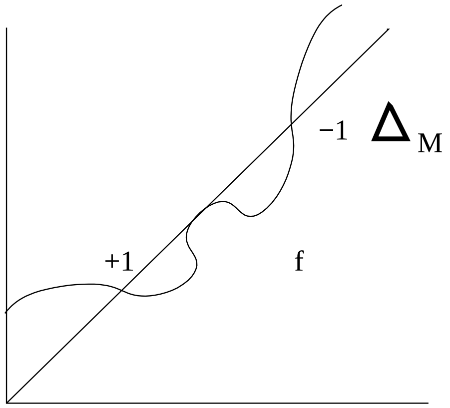

FIGURE 8

Assign to each (finite) such intersection ±1 according to the sign of $\det(I - Df_x)$. The (total) fixed point index $I(f) = I(f, M)$ is defined to be

$$I(f) := \sum_{x \in Fix f} sign(\det(I - Df_x)) \qquad \in \mathbb{Z}.$$

Next, we use rational homology groups to define the Lefschetz number. For every dimension $p \geq 0$, let $C_p(M; \mathbb{Q})$ denote the $\mathbb{Q}$-vectorspace generated by all the oriented p-simplices of M. The map f induces a linear transformation $f_{\#_p} : C_p(M; \mathbb{Q}) \to C_p(M; \mathbb{Q})$. The collection $\{C_p(M; \mathbb{Q})\}$ together with the usual boundary operator ∂ forms a *chain complex*. For each p, choose a basis $\sigma_1^p, ..., \sigma_{k(p)}^p$ for $C_p(M; \mathbb{Q})$. Then $f_{\#_p}$ can be represented by a matrix (a_{ij}) given by

$$f_{\#_p}(\sigma_j^p) = \sum_{i=1}^{k(p)} a_{ij} \sigma_i^p.$$

Define

$$\mathcal{T}r(f_\#) = \sum_{q=0} (-1)^q \text{trace}(f_{\#_q}).$$

Recall that the p-th homology group is defined to be

$$H_p(M;\mathbb{Q}) := \frac{Ker(\partial_p : C_p \to C_{p-1})}{Im(\partial_{p+1} : C_{p+1} \to C_p)}.$$

Then f induces a linear transformation

$$f_{*_p} : H_p(M;\mathbb{Q}) \to H_p(M;\mathbb{Q}).$$

Again, by choosing a basis, we can represent f_{*_p} by a matrix. The *Lefschetz number* of f is defined to be

$$L(f) := \sum_{q=0} (-1)^q \mathrm{trace}(f_{*_q}) \qquad \in \mathbb{Q}.$$

The following is the Lefschetz Fixed Point Theorem.

Theorem 5.1.

$$I(f) = L(f)$$

so that $L(f)$ is an integer. In particular, $L(f) \neq 0$ implies that $Fix f \neq \emptyset$.

Remark 5.1. The Lefschetz Fixed Point Theorem was first proved by Solomon Lefschetz (1884 - 1972) in 1923 (improved in 1926 and extended to manifolds with boundary in 1927). In 1928, Heinz Hopf (1894 - 1971) gave a different and simpler proof for selfmaps of compact polyhedra. He showed that $L(f) = \mathcal{T}r(f_{\#})$ at the chain level. Thus, when $L(f) \neq 0$, $\mathrm{trace}(f_{\#_q}) \neq 0$, i.e., σ_i^q is mapped to itself by $f_{\#_q}$ and so by the Brouwer Fixed Point Theorem, f must have a fixed point.

6. The Fixed Point Index

For our purposes, we consider fixed point indices of maps defined on open subsets of Euclidean spaces. Given an isolated fixed point $x_0 \in U \subset \mathbb{R}^n$ of a map $f : U \to \mathbb{R}^n$, the *fixed point index* $I(f, x_0) = I(f, U_0)$ is the topological degree of the map $id - f$ in a small neighborhood U_0 of x_0 not containing any other fixed points of f. The fixed point index is characterized by the following properties (axioms).

Let $f : U \to \mathbb{R}^n$ or $f : U(\subset X) \to X$ be a map where X is a polyhedron (assume that $Fix f$ is compact in U).

(1) If $I(f, U) \neq 0$ then f has a fixed point in U.

(2) (homotopy invariance) If $H : U \times [0, 1] \to X$ is a homotopy such that $\bigcup_t Fix H_t \subset U$ is compact then

$$I(H_0, U) = I(H_1, U).$$

Remark 6.1. The compactness of $\bigcup_t Fix H_t$ in U cannot be relaxed. For example, let U be the unit open disk in $X = \mathbb{R}^2$. Let $f : U \to X$ be the constant map at the origin $(0, 0)$. Define $H_t(x) = (t, 0), 0 \leq t \leq 1$. Note that $I(f, U) = 1$ but $Fix H_1 = \emptyset$ so $I(H_1, U) = 0$. Here, $\bigcup_t Fix H_t$ is not compact in U.

(3) (additivity) Suppose $U_1, ..., U_r$ are mutually disjoint open subsets of U and $Fix f \subset \bigcup U_j$. Let $f_j = f|_{U_j}$. Then,

$$I(f, U) = \sum_j I(f_j, U_j).$$

(4) (multiplicativity) Given $f : U \to X, g : V \to Y$, consider the product map

$$f \times g : U \times V(\subset X \times Y) \to X \times Y.$$

Then,

$$I(f \times g, U \times V) = I(f, U) \cdot I(g, V).$$

(5) (commutativity) Let U, V be open subsets of X, Y respectively. Given two maps $f : U \to Y$ and $g : V \to X$. Consider the maps

$$g \circ f : f^{-1}(V) \to X, \qquad f \circ g : g^{-1}(U) \to Y.$$

Then, $Fix(g \circ f)$ is homeomorphic to $Fix(f \circ g)$ and

$$I(g \circ f, f^{-1}(V)) = I(f \circ g, g^{-1}(U)).$$

We say that a map $f : U$ (open subset of X) $\to X$ is *compactly fixed* if $Fix f$ is compact in U. A homotopy $\{H_t\} : U \to X$ is *compactly fixed* if $(\bigcup_t Fix H_t)$ is compact in U.

(6) (local removability) Suppose f has an isolated fixed point x_0 such that $I(f, U_0) = 0$ for some open neighborhood U_0 of x_0 such that $Fix f \cap U_0 = \{x_0\}$. For any open neighborhood V of x_0 with $V \subset U_0$, there exists a map $g : U \to X$ compactly fixed homotopic to f such that $g \equiv f$ on $U - V$ and $Fix g = Fix f - \{x_0\}$.

Note that from the additivity property of the fixed point index, if W is an open set such that $Fix f \subset W \subset cl(W) \subset U$ then $I(f, U) = I(f|_W, W)$.

The next property is the same as the Lefschetz-Hopf Theorem.

(7) (normalization)

$$L(f) = I(f, X).$$

In particular, if f has isolated fixed points $x_1, ..., x_k$ then

$$L(f) = \sum_j I(f, x_j).$$

Remark 6.2. Properties **(1) - (5)** and **(7)** characterize the Lefschetz number. That is, any integer-valued function satisfying these properties coincides with the fixed point index.

Example 6.1. Let $f : S^n \to S^n$ be a selfmap where S^n is the unit n-sphere with $n \geq 1$. If we use cellular homology then we can decompose $S^n = \{e_0\} \cup \{\sigma^n\}$ as a union of two cells, one in dimension 0 and one in dimension n. Note that

$$H_i(S^n) = \begin{cases} 0, & \text{if } i \neq 0, n; \\ \mathbb{Z}, & \text{otherwise.} \end{cases}$$

It follows that $f_{*0} = \text{id}$ and thus $\text{trace}(f_{*0}) = 1$. The homomorphism $f_{*n} : \mathbb{Z} \to \mathbb{Z}$ is a 1×1 matrix so $\text{trace}(f_{*n}) = \deg f$. Hence,

$$L(f) = 1 + (-1)^n \deg f.$$

In general, if $f = 1_M$ or f is homotopic to the identity map 1_M then $\text{trace}(f_{*p}) = \dim H_p(M; \mathbb{Q})$ so that $L(f) = L(1_M) = \chi(M)$, the Euler characteristic of M.

Example 6.2. If $M = D^n$ which is contractible then $H_p(D^n;\mathbb{Q}) = 0$ for all $p > 0$. Thus, for any $f : D^n \to D^n$, $L(f) = 1 \neq 0$. Hence, the Brouwer Fixed Point Theorem follows from the Lefschetz Fixed Point Theorem.

7. Converse of the Lefschetz Theorem

Since $I(f, M)$ is invariant under homotopy and $I(f, M) = L(f)$, it follows that $L(f)$ is also a homotopy invariant. (Of course, $f \sim f'$ implies that $f_{*_p} = f'_{*_p}$ so that $L(f)$ is invariant under homotopy.) Therefore, the converse of the Lefschetz Fixed Point Theorem is the statement:

1. *If* $L(f) = 0$ *then there exists* f' *so that* $f \sim f'$ *and* $Fix f' = \emptyset$.

As it turns out that the converse does not hold in general. It is natural to ask for what condition under which the converse holds. From the point of view of intersection theory, we can formulate the converse problem as follows.

2. *For any map* $f : M \to M$, $1 \times f : M \to M \times M$ *is deformable into* $M \times M - \Delta_M$

The projection $p : M \times M \to M$ given by $(x, y) \mapsto x$ restricts to $p' : M \times M - \Delta_M \to M$ where $M \times M - \Delta_M = \{(x, y)|x \neq y\}$. For any $x_0 \in M, p'^{-1}(x_0) = \{(x_0, y)|y \neq x_0\} \approx M - \{x_0\}$. Then

$$p : (M \times M, M \times M - \Delta_M) \to M$$

is a *fibered pair* with typical fiber $(M, M - \{x_0\})$. Thus, deforming the graph of f into $M \times M - \Delta_M$ is equivalent to finding a section $s' : M \to M \times M - \Delta_M$ to p', i.e., $p' \circ s' = 1_M$. This setting is well suited for employing classical obstruction theory.

In order to address the insufficiency of $L(f) = 0$, we need a more subtle invariant. We shall see in the next lecture that the Nielsen number is one such number. In fact, for manifolds of dimension at least 3, the vanishing of the Nielsen number $N(f)$ is sufficient to guarantee fixed point free maps in the homotopy class of f. However, $N(f) = 0$ is not adequate in dimension 2.

8. Jiang's counter-example

This example of Jiang published in 1984 gives a negative answer to the so-called Nielsen's conjecture. In this example, every map homotopic to the given map must have at least 2 fixed points while the Nielsen number is zero.

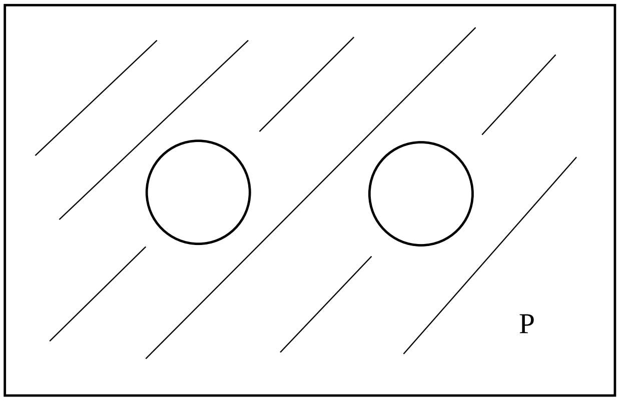

Figure 9

Consider the *pants* surface P (with boundary). Since P has the same homotopy type of the figure-eight, we define a map on the figure-eight ∞ and then extend to P.

Let $f_0 : \infty \to \infty$ be a map so that

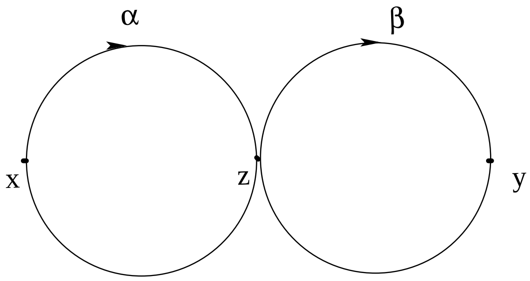

Figure 10

$f_0(\alpha) = \alpha^{-1}$ and $f_0(\beta) = \alpha^{-1}\beta^2$. Then define $f : P \to P$ to be the composite $f = i \circ f_0 \circ r$ where $r : P \to \infty$ is a retraction and $i : \infty \to P$ is the inclusion. Now, $Fix f = \{x, y, z\}$ so that $I(f, x) = 1, I(f, y) = -1$, and $I(f, z) = 0$. Taking the sum of the fixed point indices yields $L(f) = 0$. In fact, $N(f) = 0$.

The existence of a fixed point free map amounts to solving some equations involving elements in the fundamental group $\pi_1(P \times P - \Delta_P)$, also known as the *pure braid group* on P.

Lecture III - Nielsen Fixed Point Theory

We define the Nielsen number of a selfmap. A classical result of F. Wecken on the minimization of the fixed point set is outlined. We discuss computation and algebraic estimation of the Nielsen number.

9. The Nielsen number

In his pioneering work on surface automorphisms, Jakob Nielsen (1890 - 1959) introduced the notion of fixed point classes in 1927. This idea has led to a fruitful study of topological fixed point theory, now known as Nielsen fixed point theory.

Let X be a compact connected polyhedron and $f : X \to X$ a map. The fixed point set $Fix f = \{x \in X | f(x) = x\}$ is compact. Suppose $x, y \in Fix f$. We say that x and y are *Nielsen equivalent as fixed points* of f if there exists a path $\sigma : [0, 1] \to X$, $\sigma(0) = x, \sigma(1) = y$ such that σ is homotopic to $f \circ \sigma$ relative to the endpoints.

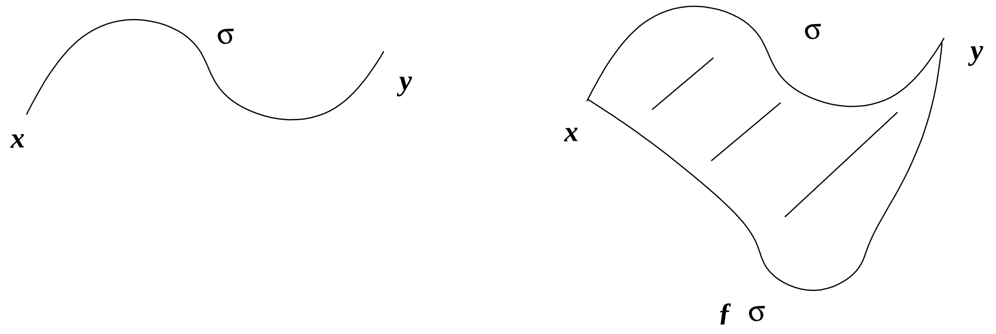

FIGURE 11

This is an equivalence relation on $Fix f$ and the equivalence classes are called *Nielsen classes* of f. Note that a Nielsen class is non-empty by definition.

Given a Nielsen class $\mathcal{F}$, there is an open neighborhood $U \supset \mathcal{F}$ such that $U \cap Fix f = \mathcal{F}$. Then $I(f, U)$ is defined. The class $\mathcal{F}$ is said to be *essential* if $I(f, U) \neq 0$. The essentiality of $\mathcal{F}$ is independent of the choice of U so that we write $I(f, \mathcal{F}) = I(f, U)$. The Nielsen number of f is defined by

$$N(f) = \#\{\mathcal{F} | I(f, \mathcal{F}) \neq 0\}.$$

Among the basic properties of $N(f)$, the following are the most important.

(1) $0 \leq N(f) \leq \# Fix f$.

By definition, $N(f)$ is a non-negative integer. Since each Nielsen class contains at least one fixed point of f, it follows that $N(f) \leq \# Fix f$.

(2) If $f' \sim f$ then $N(f') = N(f)$.

To see that the Nielsen number is a homotopy invariant, let $\{f_t\}$ be a homotopy so that $f_0 = f$ and $f_1 = f'$. Consider the map $F : X \times [0, 1] \to X \times [0, 1]$ given by $F(x, t) = (f_t(x), t)$. Note that $(x, t) \in Fix F$ iff $x \in Fix f_t$. Let $\mathcal{F}$ be the fixed point class of F containing $(x_0, 0)$. Let $\mathcal{F}_0 = \mathcal{F} \cap (X \times \{0\})$ and $\mathcal{F}_1 = \mathcal{F} \cap (X \times \{1\})$. Furthermore, write $\mathbb{F}_0 = pr_1(\mathcal{F}_0)$ and $\mathbb{F}_1 = pr_1(\mathcal{F}_1)$ where pr_1 is the projection onto X.

Observe that $\mathbb{F}_0$ is a fixed point class of f_0 and $\mathbb{F}_1$ is a fixed point class of f_1 (possibly empty).

By the homotopy invariance of the fixed point index, we have

$$I(f_0, \mathbb{F}_0) = I(f_1, \mathbb{F}_1).$$

This implies that $N(f)$ is a homotopy invariant. Unlike the Lefschetz number $L(f)$ whose non-vanishing implies existence of fixed points, $N(f)$ gives possible multiple fixed points provided we can compute $N(f)$. It is therefore natural to ask whether the Nielsen number $N(f)$ can be realized within the homotopy class of f.

The following classical result of F. Wecken makes the Nielsen number an important object of study.

THEOREM 9.1. *Suppose X is a compact connected triangulated manifold of dimension at least 3. For any $f : X \to X$, there exists a map f' homotopic to f such that $\# Fix f' = N(f)$.*

PROOF. (Sketch) (1) Deform f to f_1 such that $\# Fix f_1 < \infty$.

(2) If $x \in Fix f_1$ has index 0 then we can remove x locally, i.e., there exists $f_2 \sim f_1$ such that for any $y \in Fix f_2, I(f_2, y) \neq 0$.

(3) Suppose x and y belong to the same Nielsen class, i.e., there exists a path $C : [0,1] \to X, C(0) = x, C(1) = y$ such that $f \circ C \sim C$. Choose a contractible neighborhood U of C (may assume that U does not contain any other fixed points). (See figure below.)

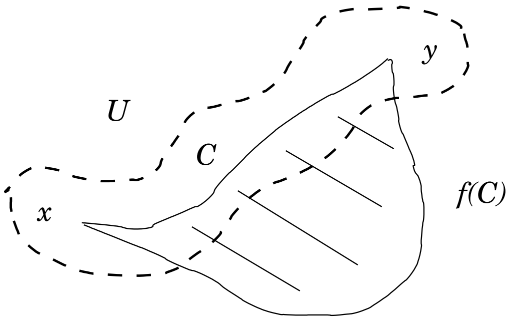

FIGURE 12

Deform f_2 to f_2' relative to $X - U$ such that $Fix f_2' = Fix f_2 - \{y\}$. (Note that the local index of f_2' at x is different from that of f_2 at the same point.) Repeating this process a finite number of times, we will arrive at a map $f' \sim f$ such that each Nielsen class of f' has exactly one fixed point and has non-zero index. □

REMARK 9.1. As a consequence of Wecken's theorem, one can prove a classical theorem of H. Hopf which aserts that a compact smooth manifold M admits a nowhere vanishing vectorfield if and only if $\chi(M) = 0$. The existence of a nonsingular vectorfiled is equivalent to deforming the identity map 1_M to be fixed point free. Since $f = 1_M$, there is only one Nielsen class so that $\chi(M) = L(1_M) = 0$ iff $N(1_M) = 0$. If $\dim M \geq 3$ then Theorem 9.1 implies that 1_M is homotopic to a fixed point free map. In fact, for the identity map, Wecken proved a similar minimality theorem with weaker assumptions so that the theorem is valid for 2-dimensional manifolds. The dimension 1 case is obvious. For more details, see Chapter 6 of [**15**] or Chapter VIII of [**1**].

The so-called Nielsen's Conjecture is that the Wecken's minimality theorem holds for surfaces. We have seen Jiang's counter-example in which there is a surface map f with $N(f) = 0$ but $\#Fixf' > 0$ for all $f' \sim f$.

Remark 9.2. Jiang's counter-example was later modified to give counter-examples of selfmaps on surfaces without boundary.

In practice, we must use algebraic techniques to compute $N(f)$. Next we present Nielsen fixed point theory from the covering space approach.

10. Covering Space Approach

Since a Nielsen class is assumed to be non-empty, it is more desirable to consider classes that may be empty. This is the covering space approach to the theory.

Let X be a compact connected polyhedron. Denote by $\eta : \tilde{X} \to X$ the universal cover of X. Suppose $f : X \to X$ is a map. Let $\tilde{f} : \tilde{X} \to \tilde{X}$ be a lift of f to the universal cover so that

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\eta"'] & \tilde{X} \arrow[d, "\eta"] \\
X \arrow[r, "f"'] & X
\end{tikzcd}$$

is commutative.

The group of deck transformations of η is simply the set of all lifts of the identity map 1_X. We denote this set by $Cov\eta$ (covering group).

Choose a basepoint $\tilde{x}_0 \in \tilde{X}$ and let $\alpha \in Cov\eta$. By the uniqueness of lifts, there exists a unique element $\tilde{\alpha} \in Cov\eta$ such that

$$\tilde{\alpha}(\tilde{f}(\tilde{x}_0)) = \tilde{f}(\alpha(\tilde{x}_0)).$$

Let the assignment $\alpha \mapsto \tilde{\alpha}$ be $\varphi : Cov\eta \to Cov\eta$. It follows that

$$\tilde{f}\alpha = \varphi(\alpha)\tilde{f}, \qquad \text{for all } \alpha \in Cov\eta.$$

If $\beta \in Cov\eta$, then

$$\varphi(\alpha)\varphi(\beta)\tilde{f} = \varphi(\alpha)\tilde{f}\beta = (\tilde{f}\alpha)\beta = \tilde{f}(\alpha\beta) = \varphi(\alpha\beta)\tilde{f}$$

which implies that $\varphi(\alpha\beta) = \varphi(\alpha)\varphi(\beta)$ so φ is a group homomorphism.

Note that if $\tilde{f}(\tilde{x}) = \tilde{x}$ then $\eta\tilde{f}(\tilde{x}) = \eta\tilde{x}$ and so $\eta\tilde{x} = f\eta(\tilde{x})$. In other words, $\eta(\tilde{x}) \in Fixf$. Thus, fixed points of lifts of f project to fixed points of f.

Example 10.1. Take $X = S^1$ and so $\tilde{X} = \mathbb{R}$. Let $f : S^1 \to S^1$ be defined by

$$f(e^{it}) = e^{i(\pi - t)}.$$

Suppose we let $\tilde{f}_k(t) = -t + k\pi$ where $k \in \mathbb{Z}$. Then $\tilde{f}_k$ is a lift of f iff $\eta\tilde{f}_k(t) = f\eta(t) = e^{i(\pi-t)}$. This in turn is equivalent to $\eta(-t + k\pi) = e^{-it}e^{ik\pi} = e^{i(\pi-t)}$ which is the same as saying that k must be odd. When $k = 1$, $Fix\tilde{f}_1 = \{\frac{\pi}{2}\}$. For $k = 5$, $Fix\tilde{f}_5 = \{\frac{5\pi}{2}\}$. However,

$$\eta Fix\tilde{f}_1 = \{e^{i\pi/2}\} = \eta Fix\tilde{f}_5$$

so different lifts may yield the same fixed points of f.

In general, if $\tilde{f}$ is a lift and $\gamma \in Cov\eta$ then

$$\eta Fix\tilde{f} = \eta Fix(\gamma\tilde{f}\gamma^{-1}).$$

This follows from the fact that $\eta\tilde{x} = \eta\gamma\tilde{x}$ and

$$\tilde{x} = \tilde{f}(\tilde{x}) \Leftrightarrow \gamma\tilde{f}\gamma^{-1}(\gamma\tilde{x}) = \gamma\tilde{x}.$$

In fact,

$$\eta Fix\tilde{f} = \eta Fix\tilde{f}' \Rightarrow \tilde{f}' = \gamma\tilde{f}\gamma^{-1}$$

for some $\gamma \in Cov\eta$.

Let $[\tilde{f}]$ denote the equivalence class containing $\tilde{f}$ by the relation

$$[\tilde{f}_1] = [\tilde{f}_2] \Leftrightarrow \tilde{f}_2 = \gamma \tilde{f}_1 \gamma^{-1}$$

for some $\gamma \in Cov\eta$. Thus different *lifting classes* yield different fixed points of f. In fact, we have

$$Fix f = \bigsqcup_{[\tilde{f}]} \eta Fix \tilde{f},$$

i.e., $Fix f$ is a disjoint union of projections of fixed points of lifts from distinct lifting classes.

11. Reidemeister Classes

By fixing a lift $\tilde{f}$, we obtain a homomorphism $\varphi : Cov\eta \to Cov\eta$ such that

$$\tilde{f}\gamma = \varphi(\gamma)\tilde{f}, \qquad \text{for all } \gamma \in Cov\eta.$$

Since every lift of f is of the form $\alpha\tilde{f}$, we ask when two lifts belong to the same lifting class. Let $\alpha, \beta \in Cov\eta$. Then

$$\begin{aligned} [\alpha\tilde{f}] = [\beta\tilde{f}] &\Leftrightarrow \beta\tilde{f} = \gamma(\alpha\tilde{f})\gamma^{-1} \qquad \text{for some } \gamma \in Cov\eta \\ &\Leftrightarrow \beta\tilde{f} = \gamma\alpha(\tilde{f}\gamma^{-1}) = \gamma\alpha\varphi(\gamma)^{-1}\tilde{f}. \end{aligned}$$

By the uniqueness of lifts, we have

$$[\alpha\tilde{f}] = [\beta\tilde{f}] \Leftrightarrow \beta = \gamma\alpha\varphi(\gamma)^{-1} \tag{11.1}$$

for some $\gamma \in Cov\eta$.

Furthermore, by choosing appropriate basepoint, φ can be identified with $f_\#$ on π_1 (here we also identify $Cov\eta$ with π_1). From now on, we write π for both $Cov\eta$ and $\pi_1(X)$.

Using (11.1), we define the *Reidemeister action* (of π on π) to be

$$\gamma \bullet \alpha \mapsto \gamma\alpha\varphi(\gamma)^{-1} \tag{11.2}$$

The orbits of the action (11.2) are called the *Reidemeister classes.* It follows from (11.1) that there is a one-to-one correspondence between the lifting classes and the Reidemeister classes.

REMARK 11.1. If we choose a different lift $\tilde{f}'$ and thus a different homomorphism φ', we get a bijection between the φ-Reidemeister classes and the φ'-Reidemeister classes so that the cardinality of such sets is constant. Let $R(\varphi)$ be the cardinality of the set of φ-Reidemeister classes. It is called the *Reidemeister number* of φ. The Reidemeister number of the map f, denoted by $R(f)$ is simply $R(\varphi)$ or the cardinality of the sset of lifting classes.

11.1. Relationship with $N(f)$. Let $\tilde{f}$ be a lift of f to the universal cover. Suppose $\tilde{x}_1, \tilde{x}_2 \in Fix\tilde{f}$ project to different fixed points x_1, x_2 respectively. Choose a path $\tilde{C} : [0,1] \to \tilde{X}$ such that $\tilde{C}(0) = \tilde{x}_1, \tilde{C}(1) = \tilde{x}_2$. Then $C := \eta \circ \tilde{C} : [0,1] \to X$ is a path from x_1 to x_2 and the path $\tilde{f} \circ \tilde{C}$ projects to the path $f \circ C$ since $\eta(\tilde{f} \circ \tilde{C}) = f \circ \eta \circ \tilde{C} = f \circ C$. In fact, the loop $\tilde{C}(\tilde{f} \circ \tilde{C})^{-1}$ projects to the loop $C(f \circ C)^{-1}$. Since $\tilde{X}$ is simply-connected, $\tilde{C}(\tilde{f} \circ \tilde{C})^{-1}$ is trivial in π_1 and thus $C(f \circ C)^{-1}$ is homotopic to the trivial loop, i.e., $f \circ C \sim C$. In other words, x_1 and x_2 are Nielsen equivalent.

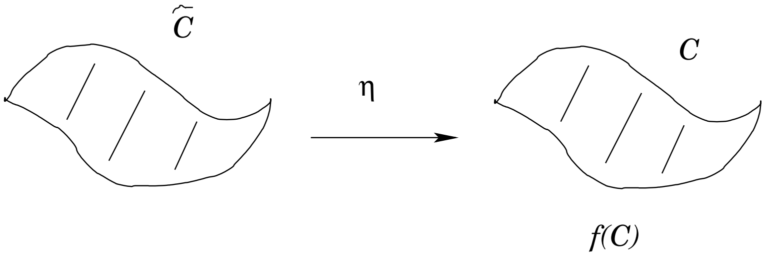

FIGURE 13

Conversely, suppose $x_1, x_2 \in Fixf$ are Nielsen equivalent, i.e., there exists a path C from x_1 to x_2 such that $f \circ C \sim C$. Let $\tilde{f}$ be a lift of f and $\tilde{x}_1 \in Fix\tilde{f}$ such that $\eta(\tilde{x}_1) = x_1$. Lift C to a path $\tilde{C}$ starting at $\tilde{x}_1$ and ending at $\tilde{x}_2$. Then $\tilde{f} \circ \tilde{C}$ projects onto $f \circ C$ which is homotopic to C. Thus, $\tilde{f} \circ \tilde{C}$ also ends at $\tilde{x}_2$ and hence $\tilde{f}(\tilde{x}_2) = \tilde{x}_2$.

If we let $\mathcal{N}(f)$ and $\mathcal{R}(f)$ denote the set of Nielsen classes and the set of Reidemeister classes respectively, then our discussion above shows that there is an injection

$$\mathcal{N}(f) \rightarrowtail \mathcal{R}(f)$$

which implies that $N(f) \leq R(f)$. Now, it is appropriate to distinguish the Nielsen (non-empty) classes from the Reidemeister classes. We call $\eta Fix \tilde{f}$ a *fixed point class* of f. It might be empty. This class corresponds to the lifting class $[\tilde{f}]$ and the Reidemeister class $[1]$ since we fix the lift $\tilde{f}$ and thereby considering the φ-Reidemeister classes. We should point out that $R(f)$ need not be finite while $N(f) < \infty$. For example, if $f = 1_X$ then any two points (fixed by 1_X) are Nielsen equivalent so $N(f) \leq 1$. On the other hand, the Reidemeister classes are simply the conjugacy classes. In particular, if $\pi_1(X)$ is abelian then $R(1_X) = |\pi_1(X)|$.

12. Computation of the Nielsen Number

We begin with the question: *How do we compute the Nielsen number?*

There are some easy cases: if $f = 1_X$ or if X is simply-connected then there is only one Nielsen class so $N(f) \leq 1$. In these situations, $L(f) = 0 \Rightarrow N(f) = 0$ or $L(f) \neq 0 \Rightarrow N(f) = 1$. Of course, if $f = 1_X$ then $L(f) = \chi(X)$. Thus, the Nielsen number does not give more information than the Lefschetz number.

In 1943, W. Franz showed that if X is the classical lens space then for any selfmap $f : X \to X$, the Nielsen classes have the same fixed point index. In fact, we have

(1) $L(f) = 0 \Rightarrow N(f) = 0$ in which case $f \sim f'$ with $Fix f' = \emptyset$;

(2) $L(f) \neq 0 \Rightarrow N(f) = R(f) = \# Coker(1 - f_{*1})$.

Let's first recall the definition of the classical lens spaces. Consider the 3-sphere

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 = 1\}.$$

Let $\mathbb{Z}_p$ be the cyclic group of order p, p odd prime. Then $\mathbb{Z}_p$ acts freely on S^3 via

$$\zeta \bullet (z_1, z_2) \mapsto (\zeta z_1, \zeta z_2)$$

where $\zeta = e^{2\pi i/p}$. The orbit space $S^3/\mathbb{Z}_p =: L_p$ is the lens space with $\pi_1(L_p) \cong \mathbb{Z}_p$. The canonical map $\eta : S^3 \to L_p$ is a regular p-fold cover.

Consider a lift $\tilde{f}$ so that

$$\begin{tikzcd} S^3 \arrow[r, "\tilde{f}"] \arrow[d, "\eta"'] & S^3 \arrow[d, "\eta"] \\ L_p \arrow[r, "f"'] & L_p \end{tikzcd}$$

is commutative. For any $\alpha \in Cov\eta$, $\eta \circ \alpha \tilde{f} = f \circ \eta$. Since $\deg \eta = p \neq 0$, it follows that $\deg(\alpha \tilde{f}) = \deg f$. Thus, $\deg(\alpha \tilde{f})$ is independent of α. By a classical theorem of H. Hopf, we conclude that $\alpha \tilde{f} \sim \tilde{f}$ for any α. This homotopy then implies that $I(f, \eta Fix \alpha \tilde{f}) = I(f, \eta Fix \tilde{f})$ so that if one class is essential all other classes are essential and $L(f) \neq 0 \Rightarrow N(f) = R(f)$. If one class is inessential then all classes are inessential, thus $L(f) = 0 \Rightarrow N(f) = 0$.

The fact that $R(f) = \#Coker(1 - f_{*1})$ follows from the fact that π_1 is abelian and thus isomorphic to H_1. More precisely, since $\pi_1(X) = H_1(X)$, $\gamma \alpha \varphi(\gamma)^{-1} = \bar{\gamma} + \bar{\alpha} - f_{*1}(\bar{\gamma}) = \bar{\alpha} + (1 - f_{*1})(\bar{\gamma})$. The Reidemeister action now becomes

$$\bar{\alpha} \mapsto \bar{\alpha} + (1 - f_{*1})(\bar{\gamma})$$

or

$$H_1(X) \to H_1(X)/\mathrm{Im}(1 - f_{*1}) = Coker(1 - f_{*1}).$$

This phenomenon lead B. Jiang (1963) to giving the so-called Jiang conditions under which all fixed point classes have the same fixed point index.

Define

$$J(X) := \{\sigma \in \pi | \sigma \alpha = \alpha \sigma \text{ for all } \alpha \in \pi \text{ and } \sigma \sim 1_{\tilde{X}}\}.$$

By definition, $J(X)$ is a central subgroup of $\pi \equiv Cov\eta \equiv \pi_1(X)$.

Remark 12.1. The group $J(X)$ is also known as the (first) Gottlieb subgroup.

A space X is called a *Jiang space* if $J(X) = \pi$. Let $\tilde{f}$ be a lift of f. For any $\alpha \in \pi$, if $J(X) = \pi$ then $\alpha \sim 1_{\tilde{X}}$ and so $\alpha\tilde{f} \sim \tilde{f}$. Hence, the fixed point classes $\eta Fix\alpha\tilde{f}$ and $\eta Fix\tilde{f}$ have the same fixed point index. Therefore, we have the following result.

Theorem 12.1. *Suppose a compact connected polyhedron X is a Jiang space. Then for any selfmap $f : X \to X$,*

$$\begin{aligned} &(1) \quad L(f) = 0 \Rightarrow N(f) = 0; \\ &(2) \quad L(f) \neq 0 \Rightarrow N(f) = R(f) = \#Coker(1 - f_{*1}). \end{aligned} \tag{12.1}$$

Jiang spaces include (1) simply-connected spaces; (2) lens spaces (or generalized lens spaces); (3) Lie groups and H-spaces; (4) coset spaces G/G_0 where G is a compact connected Lie group and G_0 is a connected subgroup.

So far, we have seen that the fundamental group plays an important role in Nielsen fixed point theory. Thus the computation must necessarily involve π_1 and thus the (universal) covering space. On the other hand, the Lefschetz number is homological in nature and is the sum of all fixed point indices of fixed point classes. Therefore, it is desirable to find some invariant that would incorporate both the homological (Lefschetz) and the geometric (Nielsen) data.

Lecture IV - Reidemeister Tracce and Computation

We introduce the Reidemeister trace which contains both $N(f)$ and $L(f)$. Some Jiang type results are presented and we discuss the finiteness of the Reidemeister number.

13. The Reidemeister Trace

In 1936, K. Reidemeister studied Nielsen fixed point theory from the algebraic viewpoint using the universal cover. We shall define a trace-like quantity which captures both $N(f)$ and $L(f)$.

Let X be a compact connected polyhedron and $f : X \to X$ a selfmap. Suppose $\tilde{f} : \tilde{X} \to \tilde{X}$ is a lift of f. First, $\tilde{X}$ inherits a simplicial structure from that of X. The simplicial chain complex $\{C_p(\tilde{X}), \partial\}$ is defined and $C_p(\tilde{X})$ is generated by the oriented p-simplices on $\tilde{X}$.

Again, we identify $\pi = \pi_1 X \equiv Cov\eta$. If $\alpha \in \pi$ and $b \in X$ then $\alpha : \eta^{-1}(b) \to \eta^{-1}(b)$. In fact, for any $\tilde{b}_1, \tilde{b}_2 \in \eta^{-1}(b)$, there exists a unique $\alpha \in \pi$ such that $\tilde{b}_2 = \alpha(\tilde{b}_1)$. We can think of $C_p(\tilde{X})$ as a collection of π-equivariant p-simplices generated by the p-simplices of X. Since X is compact, $C_p(\tilde{X})$ is a finitely generated free $\mathbb{Z}\pi$-module. Here $\mathbb{Z}\pi$ is the integral group ring of π, i.e., a typical element is of the form $\sum n_\alpha \alpha$, $n_\alpha \in \mathbb{Z}, \alpha \in \pi$ and all but a finite number of the n_α's are zero. A $\mathbb{Z}\pi$ basis for $C_p(\tilde{X})$ is a collection of p-simplices $\tilde{\sigma}_1, ..., \tilde{\sigma}_k$ each of which coresponds to a distinct p-simplex σ_i in X. By choosing a $\mathbb{Z}\pi$ basis, the chain map

$$\tilde{f}_{\#p} : C_p(\tilde{X}) \to C_p(\tilde{X})$$

has a matrix $\mathcal{M}_p$ and the trace $\mathrm{tr}\mathcal{M}_p$ is an element of the group ring $\mathbb{Z}\pi$. By fixing the lift $\tilde{f}$, we have a homomorphism $\varphi : \pi \to \pi$. We write $\mathcal{R}_\varphi[\pi]$ as the set of orbits of the Reidemeister action

$$\sigma \bullet \alpha \mapsto \sigma\alpha\varphi(\sigma)^{-1}.$$

Let $\rho : \pi \to \mathcal{R}_\varphi[\pi]$ be the orbit map which extends linearly to $\rho : \mathbb{Z}\pi \to \mathbb{Z}\mathcal{R}_\varphi[\pi]$.

The *Reidemeister trace* of f (also known as the generalized Lefschetz number) with respect to $\tilde{f}$ is given by

$$\mathfrak{L}_\pi(f, \tilde{f}) := \sum_{q=0} (-1)^q \rho \circ \mathrm{tr}\mathcal{M}_q \qquad \in \mathbb{Z}\mathcal{R}_\varphi[\pi]. \tag{13.1}$$

This "trace" is taken at the universal cover level and thus detects the fixed points of lifts of f. Suppose $\tilde{x}$ is a fixed point of $\tilde{f}$ covering the fixed point x of f. Without loss of generality, we may assume that f has finite number of fixed points each of which lies in the interior of a maximal simplex.

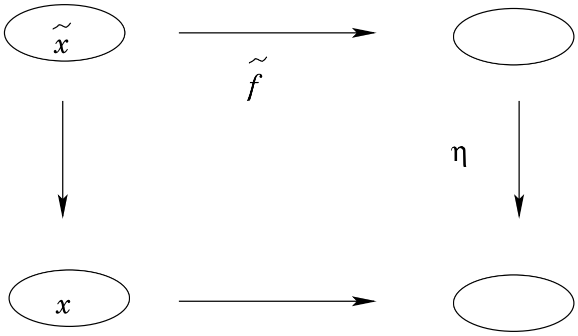

FIGURE 14

Since η is a local homeomorphism, we conclude that

$$I(\tilde{f}, \tilde{x}) = I(f, x).$$

Remember that lifts of the form $\gamma\tilde{f}\gamma^{-1}$ also has fixed points which project to x. Thus, by passing to $\mathcal{R}_\varphi[\pi]$, we would count the contribution to the trace exactly once.

Similar to the Lefschetz-Hopf Theorem, we have the following representation of the Reidemeister trace, due to Wecken.

THEOREM 13.1.

$$\mathfrak{L}_\pi(f, \tilde{f}) := \sum_{\rho \in \mathcal{R}_\varphi[\pi]} i_\rho \rho \qquad i_\rho \in \mathbb{Z}$$

where ρ is the Reidemeister class whose corresponding Nielsen class has fixed point index i_ρ.

Note that if ρ corresponds to an empty Nielsen class then $i_\rho = 0$. The Wecken representation of Theorem 13.1 implies that $N(f) = \#\{\rho | i_\rho \neq 0\}$ and $L(f) = \sum i_\rho$. Therefore, $\mathfrak{L}_\pi(f, \tilde{f})$ captures information on $N(f)$ and on $L(f)$. Furthermore, the Reidemeister trace has the same type of properties as the ordinary Lefschetz number.

14. An Example

Let's compute the Reidemeister trace for a map on the 2-torus.

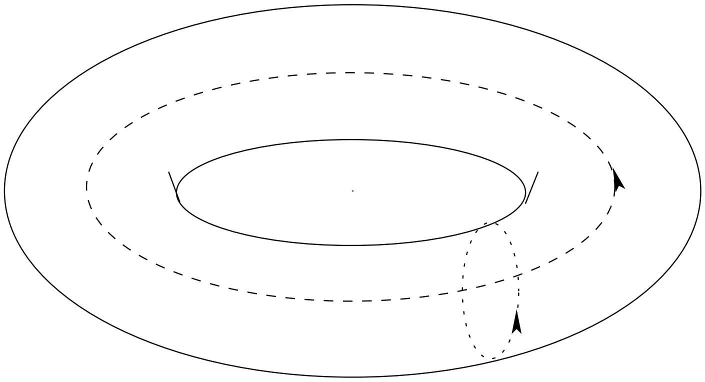

FIGURE 15

Let $T^2 = S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}^2$. The fundamental group

$$\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z} = \langle \alpha \rangle \oplus \langle \beta \rangle.$$

Let $f : T^2 \to T^2$ be a map whose induced homomorphism on π_1 is given by φ which sends $\alpha \mapsto \beta^2 \alpha^{-1}$ and $\beta \mapsto \beta \alpha^{-1}$. Consider the lift

$\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ given by the linear map $\left(\begin{smallmatrix} -1 & -1 \\ 2 & 1 \end{smallmatrix}\right)$. Consider the cells:

$$\tilde{e}_0 = (0,0) \qquad \tilde{e}_1^1 = (s,0), \tilde{e}_1^2 = (0,t)$$

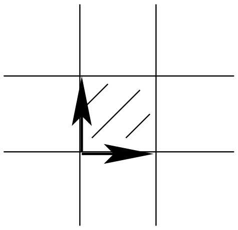

FIGURE 16

and the square $\tilde{e}_2 = (s,t), 0 \leq s \leq 1, 0 \leq t \leq 1$.

Then,

$$\tilde{f}_{\#0}(\tilde{e}_0) = \tilde{e}_0;$$
$$\tilde{f}_{\#0}(\tilde{e}_1^1) = -(\beta^2\alpha^{-1})\tilde{e}_1^1 + (1+\beta)\tilde{e}_1^2;$$
$$\tilde{f}_{\#0}(\tilde{e}_1^2) = -\alpha^{-1}\tilde{e}_1^1 + \alpha^{-1}\tilde{e}_1^2;$$
$$\tilde{f}_{\#0}(\tilde{e}_2) = (\alpha^{-1} + \alpha^{-1}\beta - \alpha^{-2}\beta^2)\tilde{e}_2.$$

To see this, consider the 1-cells:

Since $\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$,

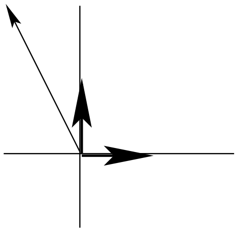

FIGURE 17

we have

$$\tilde{f}_{\#1}(\tilde{e}_1^1) = -(\beta^2\alpha^{-1})\tilde{e}_1^1 + (1+\beta)\tilde{e}_1^2.$$

Similarly, $\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$,

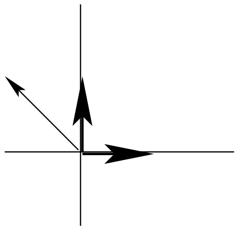

FIGURE 18

it follows that

$$\tilde{f}_{\#0}(\tilde{e}_1^2) = -\alpha^{-1}\tilde{e}_1^1 + \alpha^{-1}\tilde{e}_1^2.$$

We determine the images of the 0-cell and the 2-cell in a similar fashion.

Therefore,

$$\mathfrak{L}_\pi(f,\tilde{f}) = [1] - (-[\beta^2\alpha^{-1}] + [\alpha^{-1}]) + ([\alpha^{-1}] + [\beta\alpha^{-1}] - [\beta^2\alpha^{-2}]). \tag{14.1}$$

Now the question is: *Is* (14.1) *the Wecken representation as in Theorem 13.1?*

Next, we need to calculate the Reidemeister classes in $\mathcal{R}_\varphi[\pi]$. First, note that

$$\varphi(\alpha) = \beta^2\alpha^{-1} \quad \text{and} \quad \varphi(\beta) = \beta\alpha^{-1}.$$

Since π is abelian, if $\sigma \in \pi$ then $\sigma = \alpha^m\beta^n$ for some $m, n \in \mathbb{Z}$. Thus,

$$\begin{aligned} \sigma\alpha\varphi(\sigma)^{-1} &= \alpha^m\beta^n\alpha\varphi(\alpha^{-m}\beta^{-n}) \\ &= \alpha^m\beta^n\alpha\beta^{-2m}\alpha^m\beta^{-n}\alpha^n \\ &= \alpha^{n+1}\beta^{-2m} \end{aligned}$$

or $\sigma\varphi(\sigma)^{-1} = \alpha^n\beta^{-2m}$.

This means that the unity 1 and any element of the form $\alpha^n\beta^{\text{even}}$ are in the same Reidemeister class. Hence,

$$\mathcal{R}_\varphi[\pi] = \{[1], [\beta]\} \cong \mathbb{Z}_2.$$

Moreover,

$$[\beta^2\alpha^{-1}] = [1] = [\alpha^{-1}] = [\alpha] = [\beta^2\alpha^{-2}]$$

and

$$[\beta\alpha^{-1}] = [\beta].$$

We conclude that

$$\mathfrak{L}_\pi(f, \tilde{f}) = [1] + [\beta]$$

which implies that $L(f) = 2$ and $N(f) = 2$.

In the case of an orientable surface of genus $g > 1$, $X = \sum^g$, $\pi_1(X)$ is generated by $2g$ elements satisfying one relation. Using the Fox derivative on integral group rings, E. Fadell and S. Husseini compute the Reidemeister trace and showed that

$$\mathfrak{L}_\pi(f, \tilde{f}) = \rho(1 - \sum_i \frac{\partial f_\#(a_i)}{\partial a_i} + A)$$

where $(\frac{\partial f_\#(a_i)}{\partial a_j}) = J(\varphi)$ is the Jacobian matrix and $A \in \mathbb{Z}\pi$ such that $A(\nabla R) = \varphi(\nabla R)J(\varphi)$, R being the relation.

The most difficult part of the computation is the ability to express the Reidemeister trace in the Wecken representation, i.e., the ability to distinguish Reidemeister classes.

15. Jiang-type results

While the class of Jiang spaces contain many well-known spaces, any X with non-abelian fundamental group cannot be a Jiang space.

Since our ultimate goal is to compute $N(f)$, the effect of the Jiang condition can be achieved if we can assure that the fixed point classes are either all essential (or the same sign) or all inessential. With this in mind, B. Jiang considered the following slight generalization.

THEOREM 15.1. *Let X be a compact connected polyhedron with finite fundamental group $\pi \equiv Cov\eta$. If for each $\alpha \in \pi$, the induced homomorphism $\alpha_* : H_*(X;\mathbb{Q}) \to H_*(X;\mathbb{Q})$ is the identity then for any $f : X \to X$, (1) $L(f) = 0 \Rightarrow N(f) = 0$; (2) $L(f) \neq 0 \Rightarrow N(f) = R(f)$.*

PROOF. Let $\tilde{f}$ be a lift of f. Under the hypothesis, $(\alpha\tilde{f})_* = \tilde{f}_*$. It follows that $L(\alpha\tilde{f}) = L(\tilde{f})$. Note that the universal cover $\tilde{X}$ is also a compact connected polyhedron. The fixed point set $Fix\tilde{f}$ projects to a fixed point class $\mathbb{F}$ of f. Furthermore, $L(\tilde{f})$ is an integral multiple of $I(f, \mathbb{F})$ such that $L(\tilde{f})$ and $I(f, \mathbb{F})$ have the same sign. We conclude that all fixed point classes have fixed point indices of the same sign. □

When π is infinite and non-abelian, the general computational problem of the Nielsen number remains challenging. Recent progress has been made for coset spaces of compact connected Lie groups as well as certain $\mathcal{C}$-nilpotent spaces where $\mathcal{C}$ is the class of finite groups.

We call X a *Jiang-type* space if the conclusion of Theorem 15.1 holds for all selfmaps of X.

In 1984, D. Anosov showed that for all selfmaps $f : N \to N$ of a compact nilmanifold N, $N(f) = |L(f)|$. In fact, one can show that nilmanifolds are Jiang-type spaces. By a compact nilmanifold, we mean a coset space G/Γ where G is a connected, simply-connected nilpotent Lie group and Γ is a discrete subgroup so that G/Γ is compact. The simplest such example is that of the 3-dimensional Heisenberg manifold.

EXAMPLE 15.1. Let

$$G = \{\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right) | a, b, c \in \mathbb{R}\}$$

and

$$\Gamma = \{\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{smallmatrix}\right) | a, b, c \in \mathbb{Z}\}.$$

The group operation in G is the usual matrix multiplication. As a group, G is nilpotent and non-abelian. As a topological space, G is homeomorphic to $\mathbb{R}^3$. The coset space G/Γ is a 3-dimensional compact nilmanifold. The same construction in dimension 2 yields the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Here, $\pi_1(G/\Gamma) \cong \Gamma$ is finitely generated torsion-free nilpotent.

The Anosov theorem has since prompted many new generalizations and further investigations which play an active role in current research in topological fixed point theory.

16. Finiteness of the Reidemeister number

Jiang-type results are very useful in terms of computing the Nielsen number. But before we can establish such results, we must need to know whether $R(f)$ is finite or not for the equality $R(f) = N(f)$ to hold. This kind of question already arose concerning selfmaps of nilmanifolds. As it turns out, for a selfmap f of a nilmanifold, $L(f) \neq 0$ iff $N(f) > 0$ iff $R(f) < \infty$. The finiteness of $R(f)$ has further implication for other spaces like solvmanifolds and for Nielsen coincidence theory. In particular, if f is a selfmap of a solvmanifold and $R(f) < \infty$ then $N(f) > 0$ and $N(f) = R(f)$.

Recall that if $\varphi : \pi \to \pi$ is the homomorphism induced by a map f on the fundamental group π, then the Reidemeister number $R(f) = R(\varphi)$ is the cardinality of the set of φ-twisted conjugacy classes in π. In 1994, A. Fel'shtyn and R. Hill conjectured that for a finitely generated torsion-free group π with exponential growth, any injective endomorphism $\varphi : \pi \to \pi$ will have $R(\varphi) = \infty$. G. Levitt and M. Lustig (2000), and A. Fel'shtyn (2001) showed that the conjecture holds for automorphisms when π is Gromov hyperbolic. In 2003, D. Gonçalves and P.W. gave examples of non-Gromov hyperbolic groups of exponential growth in which automorphisms with finite Reidemeister number exist. Since then, more examples of groups have been found to have the so-called R_∞ property. A finitely generated group G is said to have the property R_∞ if for any $\varphi \in Aut(G)$, $R(\varphi) = \infty$.

The Baumslag-Solitar groups $BS(m, n)$ are important groups in low dimensional topology as well as in geometric group theory. It is well-known that $BS(2, 3)$ is non-Hopfian. Recall that the Baumslag-Solitar groups have the following presentation

$$BS(m, n) \cong \langle a, t | t^{-1} a^m t = a^n \rangle.$$

Recently, A. Fel'shtyn and D. Gonçalves showed that $BS(m, n)$ has property R_∞ except for $m = 1 = n$. Note that $BS(1, 1)$ is simply the fundamental group of the 2-torus. In fact, for any endomorphism

$\varphi : BS(1,1) \to BS(1,1)$, $R(\varphi) < \infty$ iff $\det(I - \varphi) \neq 0$. There are two types of generalization of the classical Baumslag-Solitar groups.

(1) When $m = 1$, $BS(1,n)$ are solvable. One natural generalization is the solvable group

$$\Gamma(n_1, ..., n_k) \cong \langle a, t_1, ..., t_k | t_i t_j = t_j t_i, t_i^{-1} a t_i = a^{n_i} \rangle$$

where $\{n_1, ..., n_k\}$ is a set of pairwise relatively prime integers.

(2) The fundamental group of a graph of groups (in the sense of Serre) whose vertex and edge stablizers are infinite cyclic.

It has been shown that groups of type (1) (J. Taback and P.W.) and (2) (G. Levitt) have property R_∞. Moreover, any group *quasi-isometric* to a group of type (1) or (2) has property R_∞ (J. Taback and P.W.).

Other examples of groups that have R_∞ property include fibered knot groups, certain braid groups, and certain wreath products of finitely generated abelian groups with $\mathbb{Z}$.

If $R(f) < \infty$ then it may be used to obtain (i) Jiang-type result; (ii) an upper bound for $N(f)$; or (iii) an average formula by using lifts to some finite cover.

Let us consider the following situation. Given a commutative diagram

$$\begin{tikzcd} \widehat{X} \arrow[r, "\widehat{f}"] \arrow[d, "\eta_{\widehat{X}}"'] & \widehat{X} \arrow[d, "\eta_{\widehat{X}}"] \\ X \arrow[r, "f"] & X \end{tikzcd} \tag{16.1}$$

where $\eta_{\widehat{X}} : \widehat{X} \to X$ is a finite cover. If $\hat{x}, \hat{y} \in Fix\widehat{f}$ and $\hat{x}$ and $\hat{y}$ are Nielsen equivalent as fixed points of $\widehat{f}$ then there is a path $\widehat{C}$ from $\hat{x}$ to $\hat{y}$ such that $\widehat{C} \sim \widehat{f} \circ \widehat{C}$. The projection $\eta_{\widehat{X}}(\widehat{C})$ is a path from $x = \eta_{\widehat{X}}(\hat{x})$ to $y = \eta_{\widehat{X}}(\hat{y})$ and $\eta_{\widehat{X}}(\widehat{C}) \sim f \circ \eta_{\widehat{X}}(\widehat{C})$ relative endpoints, i.e., x and y belong to the same Nielsen class of f. In general, $\eta_{\widehat{X}}(Fix\widehat{f})$ is a union of Nielsen classes of f.

If $\widehat{f}$ is a lift such that $\eta_{\widehat{X}}(Fix\widehat{f})$ is a *single* Nielsen class of f then $L(\widehat{f}) \neq 0$ implies that this Nielsen class is essential. In fact, $L(\widehat{f})$ is an integer multiple of $I(f, \eta_{\widehat{X}}(Fix\widehat{f}))$. The problem now boils down to finding a finite cover $\eta_{\widehat{X}} : \widehat{X} \to X$ so that $\eta_{\widehat{X}}(Fix\alpha\widehat{f})$ is a single class for every deck transformation α. This turns out to be related to certain strong separability properties of groups. In this case, we have an average formula

$$N(f) = \frac{1}{|Cov\eta_{\widehat{X}}|} \sum_{\alpha \in Cov\eta_{\widehat{X}}} N(\alpha\widehat{f}).$$

Bibliography

[1] Robert F. Brown, "The Lefschetz Fixed Point Theorem," Scott-Foresman, Glenview, Illinois, 1971.

[2] Robert F. Brown, Fixed point theory. *History of topology*, 271–299, North-Holland, Amsterdam, 1999.

[3] Fixed point theory and its applications. Proceedings of a conference held at the International Congress of Mathematicians, University of California, Berkeley, California, August 4–6, 1986. Edited by Robert F. Brown. Contemporary Mathematics, 72. American Mathematical Society, Providence, RI, 1988.

[4] Handbook of topological fixed point theory. Edited by Robert F. Brown, Massimo Furi, Lech Górniewicz and Boju Jiang. Springer, Dordrecht, 2005.

[5] Fixed point theory. Proceedings of a Conference held at the Université de Sherbrooke, Sherbrooke, Quebec, June 2–21, 1980. Edited by Edward Fadell and Gilles Fournier. Lecture Notes in Mathematics, 886. Springer, Berlin, 1981

[6] Daciberg L. Gonçalves and José Carlos de Souza Kiihl, "Teoria do Índice," 14° Colóquio Brasileiro de Matemática, IMPA, Rio De Janeiro, 1985.

[7] Theory of fixed points and its applications. Proceedings of the International Conference held at the University of São Paulo, São Paulo, July 26–30, 1999. Edited by Daciberg L. Gonçalves and Peter N. Wong. Topology Appl. 116 (2001), no. 1. Elsevier Science B.V., Amsterdam, 2001.

[8] Andrzej Granas and James Dugundji, "Fixed point theory," Springer Monographs in Mathematics, Springer-Verlag, New York, 2003

[9] Nielsen theory and related topics. Proceedings of the conference held in St. John's, Newfoundland, June 28–July 2, 2004. Edited by Philip R. Heath, Robert F. Brown, and Edward C. Keppelmann. Fixed Point Theory Appl. 2006, Special Issue.

[10] Nielsen theory and Reidemeister torsion. Papers from the Workshop on Dynamical Zeta Functions, Nielsen Theory and Reidemeister Torsion held in Warsaw, June 24–July 5, 1996. Edited by Jerzy Jezierski. Banach Center Publications, 49. Polish Academy of Sciences, Institute of Mathematics, Warsaw, 1999.

[11] Boju Jiang, “Lectures on Nielsen Fixed Point Theory,” Contemp. Math. **14**, Amer. Math. Soc., Providence, 1983.

[12] Topological fixed point theory and applications. Proceedings of the conference held in Tianjin, April 5–8, 1988. Edited by Bo Ju Jiang. Lecture Notes in Mathematics, 1411. Springer-Verlag, Berlin, 1989.

[13] Tsai-han Kiang, “The Theory of Fixed Point Classes,” Springer, New York, 1987.

[14] Nielsen theory and dynamical systems. Proceedings of the AMS-IMS-SIAM Summer Research Conference held at Mount Holyoke College, South Hadley, Massachusetts, June 20–26, 1992. Edited by Christopher K. McCord. Contemporary Mathematics, 152. American Mathematical Society, Providence, RI, 1993

[15] James Vick, “Homology Theory,” Academic Press, New York 1973.

[16] Peter Wong, “Fixed Point Theory,” Lecture Notes, Mini-Course, XIV Brazilian Topology Meeting, July 26 - 30, 2004. (available in both English and Portuguese at `http://abacus.bates.edu/~pwong/research/research.html`)