Rates of Convergence for a Homogenization Problem in Highly Oscillating Thin Domains

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\textbf{ABSTRACT:} We combine methods from linear homogenization theory to get error estimates for the first and second terms of the asymptotic expansion on the multiple-scale method of the solutions of the Laplace’s Equation with Neumann boundary conditions posed in an one parameter family of domains $\mathcal{R}^\epsilon \subset \mathbb{R}^2$ with highly oscillating boundary and which collapse on one dimension set as the parameter $\epsilon \to 0$.

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\section{1 INTRODUCTION}

In this work we analyze convergence properties of solutions of the family of problems

$$\begin{aligned}
-\Delta w^\epsilon + w^\epsilon &= f & &\text{in } \mathcal{R}^\epsilon \\
\frac{\partial w^\epsilon}{\partial N^\epsilon} &= 0 & &\text{on } \partial \mathcal{R}^\epsilon
\end{aligned}$$

where $\mathcal{R}^\epsilon = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in (0, 1), \ 0 < x_2 < \epsilon g(x_1/\epsilon) \}$ is a thin domain with part of its boundary highly oscillating driven by a $L$-periodic function $g \in C^3(\mathbb{R})$, $N^\epsilon$ is the unit outward normal vector field to $\partial \mathcal{R}^\epsilon$ and $f(x_1, x_2) = f(x_1) \in \mathcal{W}^{2,\infty}(0, 1)$.

It is important to notice that the amplitude and period of the oscillations are of the same order $\epsilon$, which also coincides with the order of thickness of the thin domain. This scaling makes the problem very resonant and the determination of the limiting problem not straight forward. We can cite some recent works related to rough or oscillating boundaries as (Arrieta & Bruschi, 2006) where the authors consider a oscillatory behavior of the boundary of a fixed domain (without collapsing structure) proving convergence of solutions of an elliptic equation with nonlinear boundary conditions and (Casado-Díaz et al., 2010) where the authors are concerned with a viscous fluid in a domain with small height with slightly rough boundary.

The problem (1) is a variational problem and there exists a unique solution $w^\epsilon \in H^1(\mathcal{R}^\epsilon)$ for each $\epsilon > 0$. Results recently obtained in (Arrieta & Pereira, 2010; Arrieta & Pereira, 2011; Arrieta et al., 2011) show that $w^\epsilon$ converges to a "one-dimensional" limit $w_0 \in H^1(0, 1)$ in a weak topology of $H^1$, whereas in (Arrieta, 1991) the author shows that strong convergence is actually false. For this reason, we use an appropriate \textit{corrector approach} developed by (Bensoussan et al., 1978) to obtain a kind of strong convergence. This is made introducing correctors, $\kappa^\epsilon \in H^1(\mathcal{R}^\epsilon)$, $\kappa^\epsilon = o(\epsilon)$ in $L^2(\mathcal{R}^\epsilon)$ in order to

$$\epsilon^{-1/2} \| w^\epsilon - w_0 - \kappa^\epsilon \|_{H^1(\mathcal{R}^\epsilon)} \to 0.$$  

We emphasize that the classical Average’s Theorem turn out the key ingredient to get such convergence. For convenience of the reader we statement it here

\textbf{Theorem 1.1.} Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ and $f$ be a $Y$-periodic function living in $L^p(Y)$, $1 \leq p \leq \infty$. If $f^\epsilon(x) := f(\frac{x}{\epsilon})$, a.e. in $\mathbb{R}^n$, then

$$f^\epsilon \rightharpoonup \frac{1}{|Y|} \int_Y f, \ \text{as } \epsilon \to 0,$$

weakly in $L^p(\Omega)$ if $p < \infty$ and weakly-* in $L^\infty(\Omega)$.

\section{2 THE MULTIPLE-SCALE METHOD}

Motived by the periodic nature of $\mathcal{R}^\epsilon$, we consider the representative cell

$$Y^\epsilon = \{(y, z) \in \mathbb{R}^2 \mid 0 < y < L, \ 0 < z < g(y)\},$$

and recalling that $w^\epsilon \in H^1(\mathcal{R}^\epsilon)$ is solution of (1), we look for an asymptotic expansion of the form

$$w^\epsilon(x_1, x_2) = w_0(x_1, x_1/\epsilon, x_2/\epsilon) + \epsilon w_1(x_1, x_1/\epsilon, x_2/\epsilon) + \frac{\epsilon^2}{2} w_2(x_1, x_1/\epsilon, x_2/\epsilon) + \ldots$$  

where each $w_i = w_i(x, y, z) \in H^1((0, 1) \times Y^\epsilon)$ is a $L$-periodic function in the variable $y$.

The fact that $\mathcal{R}^\epsilon$ degenerates to a line segment when $\epsilon \to 0$, suggests that $w^\epsilon$ tends not to depend on the “macroscopic” variable $x_2$. This is taken into account in the choice of $w_1$.  


Introducing the expansion (2) in the problem (1), and performing the change of variables \( x = x_1, y = x_1/\epsilon, z = x_2/\epsilon \), we obtain after some computations, that the first terms on the asymptotic expansion are given by

\[
\begin{align*}
    w_0(x, y, z) &= w_0(x), \quad w_1(x, y, z) = -X(y, z)w_0'(x), \quad w_2(x, y, z) = \theta(y, z)w_0''(x),
\end{align*}
\]

where the functions \( X, \theta \in W^{2,\infty}(Y^*) \) are solutions of the auxiliary problems

\[
\begin{align*}
    -\Delta_{y,z}X(y, z) &= 0 \quad \text{in } Y^* \\
    \frac{\partial X}{\partial N}(y, g(y)) &= -\frac{g'(y)}{\sqrt{1 + (g'(y))^2}} \quad \text{on } B_1 \\
    \frac{\partial X}{\partial N}(y, z) &= 0 \quad \text{on } B_2 \\
    X(\cdot, z) &\text{ L - periodic,}
\end{align*}
\]

and

\[
\begin{align*}
    -\text{div}_{y,z} \left( \nabla_{y,z} \theta(y, z) - \left( \begin{array}{c} X(y, z) \\ 0 \end{array} \right) \right) &= 1 - r - \partial_y X(y, z) \quad \text{in } Y^* \\
    \left( \nabla_{y,z} \theta(y, z) - \left( \begin{array}{c} X(y, z) \\ 0 \end{array} \right) \right) \cdot N &= 0 \quad \text{on } B_1 \cup B_2 \\
    \theta(\cdot, z) &\text{ L - periodic,}
\end{align*}
\]

respectively. More details can be founded in (Arrieta et al., 2011). Here \( B_0 = \{(0, z) \in \mathbb{R}^2 \mid 0 < z < g(0)\} \cup \{(L, z) \in \mathbb{R}^2 \mid 0 < z < g(L)\} \) and \( B_1 = \{(y, g(y)) \in \mathbb{R}^2 \mid 0 < y < L\} \) denote the lateral and the top parts of the boundary of \( Y^* \) respectively, \( N = (N_1, N_2) \) denotes the unit outward normal vector field to \( \partial Y^* \) and

\[
r = \frac{1}{|Y^*|} \int_{Y^*} \left\{ 1 - \partial_y X(y, z) \right\} dydz > 0,
\]

is the homogenization coefficient for the problem (1). From these computations we also can characterize \( w_0 \) as the unique solution of the one dimensional equation

\[
\begin{align*}
    -r w''_0 + w_0 &= f(x), \quad x \in (0, 1) \\
    w_0(0) &= w_0(1) = 0.
\end{align*}
\]

For details we refer the reader to (Arrieta et al., 2011; Bensoussan, 1978; Cioranescu & Paulin, 1999).

### 3 MAIN RESULTS

According to (Bensoussan, 1978) we define in the thin domain \( R' \) the first and second orders correctors respectively by

\[
\begin{align*}
    \kappa'(x_1, x_2) &= -\epsilon X \left( \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon} \right) \frac{dw_0}{dx_1}(x_1) \quad \text{and} \\
    \mu'(x_1, x_2) &= \kappa'(x_1, x_2) + \epsilon^2 \theta \left( \frac{x_1}{\epsilon}, \frac{x_2}{\epsilon} \right) \frac{d^2w_0}{dx_1^2}(x_1).
\end{align*}
\]

Notice that the functions \( X \) and \( \theta \) are originally defined in the basic cell \( Y^* \), but with some abuse we will consider them periodically extended in the \( y \) direction. Therefore we have the following estimate

\[
\|X\|_{L^2(Y^*)}^2 \leq \sum_{k=1}^{1/\epsilon} \epsilon^2 \int_{Y^*} |X(y, z)|^2 dydz \leq \epsilon/L \|X\|_{L^2(Y^*)}^2.
\]

**Theorem 3.1.** Let \( w^\epsilon \in H^1(R') \) and \( w_0 \in H^1(0, 1) \) be solutions of the problem (1) and (5) respectively. Then

i) \( \epsilon^{-1/2} \|w^\epsilon - w_0 - \kappa'\|_{H^1(R')} = o(1), \) as \( \epsilon \to 0; \)

ii) \( \epsilon^{-1/2} \|w^\epsilon - w_0 - \mu'\|_{H^1(R')} = O(1/\epsilon), \) as \( \epsilon \to 0. \)

Consequently,

\[
\epsilon^{-1/2} \|w^\epsilon - w_0 - \kappa'\|_{H^1(R')} = O(1/\epsilon), \quad \text{as } \epsilon \to 0.
\]

**Proof (Sketch):** Considering in \( H^1(R') \) and in \( L^2(R') \) the inner products \( a_\epsilon(u, v) := \epsilon^{-1} \int_{R'} (\nabla u \cdot \nabla v + uv) dx \) and \( (u, v)_\epsilon = \epsilon^{-1} \int_{R'} uv dx \) respectively, we have that

\[
a_\epsilon(\varphi, w^\epsilon) = (\varphi, f)_\epsilon, \quad \forall \varphi \in H^1(R').
\]

Since \(-w_0 - \kappa' \in H^1(R') \) (and by symmetry of \( a_\epsilon \) we obtain that

\[
\epsilon^{-1} \|w^\epsilon - w_0 - \kappa'\|_{H^1(R')} = (w^\epsilon - 2w_0 - \kappa', f)_\epsilon + a_\epsilon(w_0 + \kappa', w_0 + \kappa').
\]
As noticed in (Arrieta et al., 2011) one has that \( \|w^\varepsilon - w_0\|_{L^2(\mathcal{R}^c)} \to 0 \), therefore
\[
(w^\varepsilon - w_0, f) \leq \max\{g\} \|w^\varepsilon - w_0\|_{L^2(\mathcal{R}^c)} \|f\|_{L^2(0,1)} \to 0, \quad \varepsilon \to 0.
\]

Due to (6) we have that \((\kappa^\varepsilon, f) \leq \frac{1}{L} \max\{g\} \|X\|_{L^2(Y^*)} \|w_0^\varepsilon\|_{L^2(0,1)} \to 0 \), and
\[
(w_0, f) = \int_0^1 w_0(x_1)g(x_1/\varepsilon)f(x_1) \, dx_1 = \frac{1}{L} \int_0^1 g(s) \, ds \int_0^1 w_0(x_1)f_0(x_1) \, dx_1 \quad \text{as} \quad \varepsilon \to 0.
\]

Observing that \(\varepsilon^{-1} \int_{\mathcal{R}^c} w_0^2(1 - \partial_y X(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})) \, dx_1 \, dx_2 = \int_0^1 \int_0^1 \int_0^{\theta} w_0^2(1 - \partial_y X(\frac{x_1}{\varepsilon}, z)) \, dz \, dx_1 \),
and that \(\int_0^{\theta} w_0^2(1 - \partial_y X(\frac{x_1}{\varepsilon}, z)) \, dz \) is a \(L\)-periodic function, it follows from the identity
\[
a_w(w_0 + \kappa^\varepsilon, w_0) = \varepsilon^{-1} \int_{\mathcal{R}^c} w_0^2(1 - \partial_y X(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})) \, dx_1 \, dx_2 + \varepsilon^{-1} \int_{\mathcal{R}^c} |w_0|^2 \, dx_1 \, dx_2
\]
\(- \varepsilon^{-1} \int_{\mathcal{R}^c} \left( X(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) w_0 \right) w_0^2 \, dx_1 \, dx_2 + \varepsilon X(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \, dx_1 \, dx_2
\]
that \(a_w(w_0 + \kappa^\varepsilon, w_0) \to \frac{1}{L} \int_0^L g(s) \, ds \int_0^1 (r w_0^2 + w_0^2) \, dx_1 \). Moreover, similar estimates to (6) applied for \(\partial_y X\) and \(\partial_x X\) imply \(a_w(w_0 + \kappa^\varepsilon, \kappa^\varepsilon) \to 0\). Therefore by (8) we have
\[
\varepsilon^{-1} \|w^\varepsilon - w_0 - \kappa^\varepsilon\|_{H^1(\mathcal{R}^c)} \xrightarrow{\varepsilon \to 0} \frac{1}{L} \int_0^L g(s) \, ds \left( - \int_0^1 w_0 f \, dx_1 + \int_0^1 (r w_0^2 + w_0^2) \, dx_1 \right) = 0.
\]

In order to prove (ii), we denote \(\phi^\varepsilon = w^\varepsilon - w_0 - \mu^\varepsilon\). By Chain’s Rule, (3) and (4), it follows that
\[
\sum_{i=1}^2 \partial_{x_i} \phi^\varepsilon = \sum_{i=1}^2 \partial_{x_i} w^\varepsilon = w_0'' + 2 \partial_y X w_0'' + (1 - r - 2 \partial_y X) w_0'' + c \partial_y''(X - 2 \partial_y \theta) - \varepsilon^2 \theta w_0^{(4)}.
\]

Consequently, from (1) and (5) and after some calculations we obtain
\[
- \sum_{i=1}^2 \partial_{x_i} \phi^\varepsilon + \phi^\varepsilon = -\varepsilon (w_0'') (X - 2 \partial_y \theta) - X w_0'' \varepsilon^2 \theta \left( w_0'' - w_0^{(4)} \right), \quad \text{a.e. in} \ \mathcal{R}^c.
\]

On the boundary \(\partial \mathcal{R}^c\), we have by the identity \(N^c(x_1, x_2) = N(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})\) and boundary conditions from (3) and (4) that
\[
\partial \phi^\varepsilon \overline{\partial N^c} = \partial w^\varepsilon \overline{\partial N^c} - \nabla x_1, x_2 (w_0 - \varepsilon X w_0' + \varepsilon^2 \theta w_0'') \cdot N^c
\]
\[
= w_0' \left( \partial X \overline{\partial N^c} - N_1 \right) + \varepsilon w_0'' \left( \partial X \overline{\partial N^c} - N_1 \right) - \varepsilon^2 \theta w_0'' N_1 = -\varepsilon^2 \theta w_0'' N_1.
\]

Thus, the function \(\phi^\varepsilon\) satisfies the following boundary value problem
\[
\begin{cases}
-\Delta \phi^\varepsilon + \phi^\varepsilon = \varepsilon F^\varepsilon \quad \text{in} \ \mathcal{R}^c \\
\partial \phi^\varepsilon \overline{\partial N^c} = \varepsilon^2 H^\varepsilon N_1^c \quad \text{on} \ \partial \mathcal{R}^c
\end{cases}
\]
(10)
where
\[
F^\varepsilon(x_1, x_2) = -\left( w_0''(x_1) \left( X(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) - 2 \partial_y \theta (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) \right) - X (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) w_0''(x_1) \right)
\]
\[
-\varepsilon (\theta (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) (w_0''(x_1) - w_0^{(4)}(x_1))) \quad \text{a.e. in} \ \mathcal{R}^c,
\]
and
\[
H^\varepsilon(x_1, x_2) = -\theta (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}) w_0''(x_1) \quad \text{a.e. in} \ \partial \mathcal{R}^c.
\]
(12)

Taking \(\phi^\varepsilon\) as test function in the variational formulation of (10) we obtain
\[
\varepsilon^{-1} \|\phi^\varepsilon\|_{H^1(\mathcal{R}^c)} = |a_w(\phi^\varepsilon, \phi^\varepsilon)| \leq \|\phi^\varepsilon\|_{L^2(\mathcal{R}^c)} \|F^\varepsilon\|_{L^2(\mathcal{R}^c)} + \varepsilon \|\phi^\varepsilon\|_{L^2(\mathcal{R}^c)} \|H^\varepsilon N_1\|_{L^2(\partial \mathcal{R}^c)}.
\]
(13)
To get sharp estimates for \(E^\varepsilon\) and \(H^\varepsilon\), we recall that \(f\) is a smooth function and therefore by classical regularity results we can conclude that the solution \(w_0\) of the homogenized problem is smooth enough to guarantee that its derivatives up to the fourth order are in \(L^\infty(0,1)\). The same is true for \(X, \theta \in H^1(Y^*)\).
Similarly to (6) we can get that \( \|\theta\|_{L^2(R^*)} \leq \sqrt{\epsilon/L} \|\theta\|_{L^2(Y^*)} \) and \( \|\partial_x \theta\|_{L^2(R^*)} \leq \sqrt{\epsilon/L} \|\theta\|_{L^2(Y^*)} \). Therefore, it is clear from (11) that there exists a constant \( K_0 = \frac{K_0}{\int X\|\theta\|_{L^2(Y^*)}^2 \|\partial_y \theta\|_{L^2(Y^*)}^2} \), \( \|u_0\|_{L^\infty(0,1)}, \|u'_0\|_{L^\infty(0,1)}, \|u''_0\|_{L^\infty(0,1)}, \|u'''_0\|_{L^\infty(0,1)}, \|u^{(4)}_0\|_{L^\infty(0,1)} \) (independent of \( \epsilon \)) such that

\[
\|F\|_{L^2(R^*)} \leq K_0 \epsilon^{1/2}. \tag{14}
\]

Now, let us to decompose the boundary of \( R^* \) as \( \partial_x R^* = \{ (x_1, \epsilon g(x_1/\epsilon)) \mid 0 < x_1 < 1 \} \), \( \partial_y R^* = \{ (x_1, 0) \} \mid 0 < x_1 < 1 \) and \( \partial Y^* = \{ (0, x_2) \} : 0 < x_2 < \epsilon g(0) \cup \{ (1, x_2) \} : 0 < x_2 < \epsilon g(1/\epsilon) \} \).

Thus

\[
\|H^* N'_{1}\|_{L^2(\partial R^*)}^2 \leq \|w'''_{0}\|_{L^\infty(0,1)} \int_{\partial R^*} |\theta\left(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}\right)|^2 dS \\
\leq \|w'''_{0}\|_{L^\infty(0,1)} \left( \int_{0}^{L} |\theta\left(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon}\right)|^2 dS + \int_{0}^{1} |\theta(\frac{x_1}{\epsilon}, 0)|^2 dS \right) \\
\leq K_1 \left( \sum_{k=1}^{1/sL} \epsilon \int_{0}^{L} |\theta(y, g(y))|^2 dy + \sum_{k=1}^{1/sL} \epsilon \int_{0}^{1} |\theta(y, 0)|^2 dy \right) \leq K_1 \frac{1}{L} \|\theta\|_{L^2(\partial Y^*)}^2,
\]

where \( K_1 = \|(1 + g')^{1/2} \|_{L^\infty(0,1)} \|w'''_{0}\|_{L^\infty(0,1)} \) is independent of \( \epsilon \), and we have used of the periodicity of \( \theta \) to get the integral over \( \partial R^* \) equal to 0.

We now have all the ingredients to estimate \( \|\phi''\|_{H^1(R^*)} \). Due to (14) and (15) we get from (13) that

\[
\epsilon^{-1} \|\phi''\|_{H^1(R^*)}^2 = \epsilon^{1/2} K_0 \|\phi''\|_{L^2(R^*)} + \epsilon K_2 \|\phi''\|_{L^2(\partial R^*)}.
\]

To conclude, notice that we can find a constant \( K \), independent of \( \epsilon \), such that

\[
\|\varphi\|_{L^2(\partial R^*)} \leq K \epsilon^{-1/2} \|\varphi\|_{H^1(R^*)}, \quad \forall \phi \in H^1(R^*),
\]

which proof can be found in (Cioranescu & Paulin, 1999; Lions, 1998). This remark, combined with (16), lead to (ii).

\[\Box\]

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