Describing Fermi acceleration with a scaling approach: The Bouncer model revisited

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Abstract

The behavior of average velocities on a dissipative version of the classical bouncer model is described using scaling arguments. The description of the model is made by use of a two-dimensional nonlinear area contracting map. Our results reveal that the model experiences a transition from limited to unlimited energy growth as the dissipation vanishes.

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1. Introduction

The phenomenon of Fermi acceleration has intrigued physicists for a long time [1]. It consists basically in a mechanism in which a classical particle acquires unbounded energy from collisions with a heavy moving wall [2]. One of the interesting theoretical questions on this problem lies in considering whether the Fermi acceleration can result from the nonlinear dynamics itself, as for example from collisions of a classical particle of mass $m$ with a wall, periodically time varying, and therefore in the complete absence of any imposed random assumption. It is worth stressing that there is not a unique answer to this question and the mechanism depends on the properties of the model studied. For the one-dimensional Fermi accelerator model [3–8], which consists of a classical particle confined in and bouncing between two infinitely heavy walls (where one of them is fixed and the other one is periodically oscillating), it is known that the phase space has invariant spanning curves which are responsible for limiting the energy gain of the bouncing particle. On the other hand, for a similar model [9] in which the returning mechanism for the next collision with the periodically moving wall is due only to the gravitational field [10–13], it is possible to show that, for appropriate initial conditions as well as control parameters [14], the unlimited energy growth is observed. Note however that these two examples are one-dimensional time dependent problems. For two-dimensional time varying billiards [15] the phenomenon of unlimited energy growth, when the boundaries are moving in time, depends basically on the structure of the phase space of the corresponding version of the billiard with static boundary. Thus, as conjectured by Loskutov, Ryabov and Akinshin [16], the regular dynamics for a fixed boundary implies a

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bound to the energy gained by the bouncing particle, but the chaotic dynamics of a billiard with a fixed boundary is a sufficient condition for Fermi acceleration in the system when a boundary perturbation is introduced. Such a conjecture was confirmed in many billiards problems [17–19]. However, it was shown recently that such a conjecture seems to fail for a time varying oval billiard when the breathing case is considered [20].

In this work, we revisit a dissipative version of the classical bouncer model seeking to understand and describe some of its dynamical properties. The model consists of a classical particle suffering inelastic collisions with a periodically moving wall in the presence of a constant gravitational field \( g \). The dissipation is introduced via inelastic collisions of the particle with the moving wall. This implies that there is a restitution coefficient \( \alpha \in [0, 1] \) that governs the dissipation so that, at each collision, the particle experiences a fractional loss of energy. The limit of \( \alpha = 1 \) recovers the non-dissipative case. Since in the complete absence of dissipation this model yields unlimited energy growth, our main focus in this letter is on checking whether the phenomenon of Fermi acceleration can be observed in the presence of dissipation.

This paper is organized as follows. In Section 2 we discuss all the details needed for the mapping construction. We also discuss our numerical results in the light of a scaling approach. Concluding remarks are drawn in Section 3.

2. The model, the mapping and numerical results

As is usual in the literature, the model is described via a two-dimensional mapping for the variables \((v_n, t_n)\), where \(v_n\) and \(t_n\) are the corresponding velocity of the particle and the time immediately after the \(n\)th collision with the moving wall. We assume that the motion of the moving wall is given by \(y(t) = \epsilon \cos(\omega t)\), where \(\epsilon\) denotes the amplitude of oscillation and \(\omega\) is the corresponding frequency of oscillation. Before writing down the equations, let us first discuss an approximation used, the so called simplified version, and a basic difference of its dynamics from that of the complete version. In the complete version, the motion of the moving wall is taken into account so that the particle can experience multiple hits with the moving wall without leaving the collision zone (\(y \in [-\epsilon, \epsilon]\)). The multiple hits are observed only for the regime of low energy \((V < 2\epsilon)\) and they do not happen in the simplified version. Therefore, as proposed previously by Lichtenberg and Lieberman [6], the simplified version consists in considering that the wall is fixed but that, after the collision, the particle suffers an exchange of energy and momentum as if the wall were moving; thus multiple hits no longer happen in the simplified version. Such an approximation also retains the nonlinearity of the problem. This is a common approach used to speed up the numerical simulations [21–24] since no transcendental equations [25] need to be solved, as they must be in the complete model [5,26].

We have to obtain a mapping \(T(v_n, t_n) = (v_{n+1}, t_{n+1})\). We suppose that at the time \(t = t_n\) the particle is at the position \(y = 0\) with velocity \(v = v_n > 0\). After considering the time spent either in the upward or in the downward direction, taking into account the effects of the inelastic collision and defining the following dimensionless variables: 

\[
V_n = v_n w / g, \quad \epsilon = \epsilon w^2 / g \quad \text{and measuring the time as } \phi_n = w t_n, \quad \text{we obtain that the mapping is written as}
\]

\[
T : \begin{cases} 
V_{n+1} = |\alpha V_n - (1 + \alpha) \epsilon \sin(\phi_{n+1})| \\
\phi_{n+1} = \phi_n + 2V_n \mod(2\pi).
\end{cases}
\]

We emphasize that in the simplified version, non-positive velocities are forbidden because they are equivalent to the particle moving beyond the wall. In order to avoid such problems, if after the collision the particle has a negative velocity, we inject it back with the same modulus of velocity. This procedure is effected by the use of a modulus function. Note that the velocity of the particle is reversed by the modulus function only if, after the collision, the particle remains traveling in the negative direction. The modulus function has no effect on the motion of the particle if it moves in the positive direction after the collision. This mapping is area contracting since 

\[
\det J = \alpha \text{sign}[\alpha V_n - (1 + \alpha) \epsilon \sin(\phi_{n+1})],
\]

where the function \(\text{sign}(u) = 1\) if \(u > 0\) and \(\text{sign}(u) = -1\) if \(u < 0\), and \(J\) is the Jacobian matrix.

In the non-dissipative version, the phenomenon of unlimited energy growth is observed for the condition that matches \(\epsilon > K_{\text{eff}}/4 \approx 0.2428\ldots\) with \(K_{\text{eff}} \approx 0.971\ldots\) which characterizes a transition from local to global chaos for the standard mapping (see Refs. [14,26] for a full discussion), which is also our region of interest. To illustrate some consequences introduced by the restitution coefficient, Fig. 1 shows the behavior of two initial conditions iterated \(10^4\) times. We can see that, even for a long time, the orbits stay limited to a finite region of the phase space and no unbounded energy is observed. The control parameters used in the construction of Fig. 1 are \(\epsilon = 10\) and:
Fig. 1. (Color online.) Evolution of a chaotic orbit on the phase plane $V \times \phi$ for the control parameter $\epsilon = 10$ and: (a) $\alpha = 0.99$ and (b) $\alpha = 0.999$. Convergences of the positive Lyapunov exponents, averaged over six different orbits, are shown in (c) and (d) corresponding respectively to the orbits plotted in (a) and (b).

(a) $\alpha = 0.99$ and (b) $\alpha = 0.999$. Fig. 1(c), (d) are the corresponding positive Lyapunov exponents [27] for Fig. 1(a), (b), respectively.

As shown in Fig. 1, it is expected that the average velocity grows until approaching a regime of saturation, since no Fermi acceleration is observed. This is indeed true, as shown in Fig. 2(a). We concentrate now on the evolution of the deviation of the average velocity $\omega$ [28] which is sometimes called the second momentum of $\overline{V}$. We stress that $\omega$ has the same properties as the average velocity. This implies that the same conclusions as were obtained for $\omega$ can be immediately extended to the average velocity. In order to obtain $\omega$, we must first obtain the average velocity along the orbit for a single initial condition, namely

$$\overline{V}_j(n, \epsilon) = \frac{1}{n} \sum_{i=1}^{n} V_{i,j},$$

where the index $j$ denotes a sample in an ensemble of $M$ different initial conditions. After that, the deviation of the average velocity is defined as

$$\omega(n, \epsilon) = \frac{1}{M} \sum_{j=1}^{M} \sqrt{\overline{V}^2_j(n, \epsilon) - \overline{V}_j^2(n, \epsilon)}.$$

We consider in our simulations an ensemble of $M = 5 \times 10^3$ different initial conditions. Let us now discuss the procedure used to obtain the initial conditions. The procedure becomes important because we want avoid higher values for the initial velocity. This is mainly because the initial velocity is also a scaling variable (see Ref. [21] for a full discussion and application in the non-dissipative Fermi accelerator model). Thus, for a fixed pair of control parameters ($\epsilon, \alpha$) we select and evolve an initial condition that leads to the dynamics being chaotic. Along the chaotic orbit, each time the particle’s velocity acquires a value $V_n < 2\epsilon$, the corresponding canonical variables $(V_n, \phi_n)$ are collected and used as a sample in the ensemble of $M$ initial conditions. The choice of $V_n < 2\epsilon$ is related to the fact that the particle has low energy as compared to the moving wall velocity, as was defined previously by Lichtenberg.
Fig. 2. (Color online.) (a) Behavior of the average velocity for chaotic orbits. The control parameters used are $\epsilon = 10$ and $\alpha = 0.999$. (b) Evolution of the deviation of the average velocity as a function of the iteration number. The control parameters used in the construction of the three curves are $\epsilon = 10$, $\alpha = 0.998$, $\alpha = 0.9998$ and $\alpha = 0.99998$.

and Lieberman [3]. The procedure of keeping small velocities is stopped only when the ensemble of $M = 5 \times 10^3$ initial conditions is totally filled. Moreover, it is important to say that this procedure warrants a good distribution for the pairs of initial conditions $V, \phi$ for the intervals $V \in (0, 2\epsilon)$ and $\phi \in [0, 2\pi)$.

Fig. 2(b) shows the behavior of $\omega$ for three different damping coefficients, as labelled in the figure. We can see that the deviation of the average velocity starts growing as the iteration number evolves and then suddenly it bends towards a regime of saturation, $\omega_{\text{sat}}$. It is easy to see that the changeover from growth to the saturation is marked by a crossover iteration number, $n_x$. The different damping coefficients yield the saturation occurring at different values. On the basis of the behavior shown in Fig. 2(b), we can suppose that $\omega$ behaves as

$$\omega \propto n^\beta, \quad (4)$$

where $\beta$ is a critical exponent. Eq. (4) is valid for $n \ll n_x$. The second supposition is

$$\omega_{\text{sat}} \propto (1 - \alpha)^\gamma, \quad (5)$$

where $\gamma$ is also a critical exponent and Eq. (5) is valid for $n \gg n_x$. Finally, the crossover iteration number is given by

$$n_x \propto (1 - \alpha)^z, \quad (6)$$

and $z$ is a dynamical exponent.

Considering the three scaling hypotheses, we can now describe the deviation of the average velocity formally in terms of a scaling function of the type

$$\omega((1 - \alpha), n) = l\omega(l^a(1 - \alpha), l^b n), \quad (7)$$

where $l$ is a scaling factor, $a$ and $b$ are scaling exponents. It must be emphasized that $a$ and $b$ can be related to the critical exponents $\gamma$, $\beta$ and $z$. Choosing the scaling factor as $l = (1 - \alpha)^{-1/a}$, we can rewrite Eq. (7) as

$$\omega((1 - \alpha), n) = (1 - \alpha)^{-1/a} \omega_1((1 - \alpha)^{-b/a} n), \quad (8)$$
where the function $\omega_1((1-\alpha)^{-b/a}n) \equiv \omega(1, (1-\alpha)^{-b/a}n)$ is assumed to be constant for $n \gg n_x$. An immediate comparison of Eq. (8) with Eq. (5) allows us to conclude that $\gamma = -1/a$. Choosing now $l = n^{-1/b}$, we obtain

$$\omega((1-\alpha), n) = n^{-1/b} \omega_2(n^{-a/b}(1-\alpha)), \quad (9)$$

and the function $\omega_2(n^{-a/b}(1-\alpha)) \equiv \omega(n^{-a/b}(1-\alpha), 1)$, which we assume as constant for $n \ll n_x$. Comparing the Eqs. (9) and (4), we obtain $\beta = -1/b$. Moreover, given the two different expressions of the scaling factor $l$, we have

$$z = \frac{\gamma}{\beta}. \quad (10)$$

Note that the scaling exponents are all determined if the critical exponents $\alpha$ and $\beta$ are obtained numerically. Fig. 3 shows the behavior of (a) $\omega_{\text{sat}} \times (1-\alpha)$ and (b) $n_x \times (1-\alpha)$. We stress that the saturation values for $\omega$ are obtained via extrapolation since even after almost $10^3 n_x$, the saturation value for $\omega$ has not been reached yet. Applying power law fits for Fig. 3(a) and (b) we obtain $\gamma = -0.4992(9)$ and $z = -0.982(2)$. Averaging the exponent $\beta$ for many different time series, we have $\beta = 0.500(2)$. Since the dynamical exponent $z$ can also be obtained from Eq. (10), from evaluation of the numerical values of $\gamma$ and $\beta$, we obtain $z = -0.998(3)$. This result is in good agreement with the numerical result obtained in Fig. 3(b).

Given now that the values of the critical exponents are obtained, the three scaling hypotheses can be checked. Fig. 3(c) shows the collapse for three different curves of the deviation of the average velocity generated from different values of the control parameters (as labelled in the figure) onto a single and universal plot. The collapse obtained in Fig. 3(c) reinforces that the scaling suppositions are indeed correct.

Let us now discuss some of the implications of the critical exponents $\beta$, $\gamma$ and $z$. The exponent $\beta \approx 0.5$ obtained for the initial growth of $\omega$ assumes the same value of the exponent which characterizes the dispersion of a random particle [29]. This result is an indication that the chaotic motion exhibits similar behavior of a stochastic signal. The critical exponents $\gamma$ and $z$ play the most important role in the Fermi acceleration since they characterize the behavior of $\omega_{\text{sat}}$ and $n_x$ near the criticality. Thus, as shown in Fig. 3(a), the exponent $\gamma \equiv -0.5$ yields a divergence of the deviation of the average velocity for the limit of $\alpha \to 1$. Since no saturation for $\omega$ is observed, the crossover iteration number $n_x$ also diverges in the limit of $\alpha \to 1$ given that $z \approx -1$. The knowledge of these three critical exponents
allows one to identify to what class of universality the system belongs. For example, the dissipative bouncer model does not belong to the same class of universality as the one-dimensional Fermi accelerator model since the critical exponents for that model are $z = -1$, $\beta = 0.5$ and $\gamma = 0.5$. However and despite the differences, the one-dimensional Fermi accelerator model belongs to the same class of universality as the periodically corrugated waveguide.

3. Final remarks

In summary, we have studied a dissipative version of a simplified classical bouncer model. Our results for the average velocity and for the deviation of the average velocity confirm that the model experiences a transition from limited to unlimited energy growth when the damping coefficient $\alpha \to 1$.

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References

[15] A billiard is a dynamical system in which a classical point-like particle moves within a certain region $Q$ with a piecewise smooth boundary $\partial Q$ under the condition of equality between the angles of incidence and reflection.
[25] The solutions of the transcendental equations give the time at which the particle collides with the moving wall. The equations are constructed by matching the condition that, at the instant of collision, the particle’s position is the same as that of the time varying wall.