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RELAXATION AND TRANSIENTS IN A TIME-DEPENDENT LOGISTIC MAP

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We study the one-dimensional logistic map with control parameter perturbed by a small periodic function. In the pure constant case, scaling arguments are used to obtain the exponents related to the relaxation of the trajectories at the exchange of stability, period-doubling and tangent bifurcations. In particular, we evaluate the exponent z which describes the divergence of the relaxation time τ near a bifurcation by the relation $\tau \sim |R - R_c|^{-z}$. Here, R is the control parameter and R_c is its value at the bifurcation. In the *time-dependent* case new attractors may appear leading to a different bifurcation diagram. Beside these new attractors, complex attractors also arise and are responsible for transients in many trajectories. We obtain, numerically, the exponents that characterize these transients and the relaxation of the trajectories.

Keywords: Logistic map; critical dynamics.

1. Introduction

The one-dimensional logistic map has been studied extensively [May, 1976; Crutchfield & Farmer, 1982; Grebogi *et al.*, 1982; Cavalcante & Leite, 2000], chiefly because it describes the typical behavior of many dissipative dynamical systems modeled by nonlinear differential equations. It also has many applications in physics, biology, electronics and other fields.

Particular attention has been paid to the qualitative change of the asymptotic behavior of the trajectories as the control parameter is changed. It is well known that the logistic map exhibits cascades of period-doubling bifurcations leading to chaos [Feigenbaum, 1978; Collet & Eckmann, 1980], tangent bifurcations giving rise to periodic windows (also called subductions [Grebogi & Ott, 1983]), intermittent behavior [Hirsch *et al.*, 1982], and crisis events (boundary, interior and merging chaotic bands crisis [Grebogi *et al.*, 1982; Grebogi & Ott, 1983]). The period-doubling cascades in one-dimensional maps have universal scaling laws [Feigenbaum, 1978; Collet & Eckmann, 1980] both in parameter and phase space. Scaling laws also characterize the intermittent behavior at tangent bifurcations.

Our main goal in this paper is to investigate the effect of a small periodic perturbation of the control parameter of the logistic map (also called parametric perturbation [Rossler *et al.*, 1989; Saratchandran *et al.*, 1996]), on the scaling laws and exponents related to relaxation and transient phenomena.

The logistic map considered in this work is defined by

$$X_{n+1} = F(X_n) = R_n X_n (1 - X_n) \tag{1}$$

$$R_n = R + \varepsilon f(n) \tag{2}$$

where ε is small and $f(n)$ is a periodic function. For $\varepsilon = 0$, the static case, we develop a scaling theory to obtain the exponent z of the relaxation time, for trajectories in the vicinity of the exchange of stability and period-doubling bifurcations. For $\varepsilon \neq 0$, called the time-dependent case, the effective control parameter is periodic. In this case, we obtain the fixed points and study their stability. Typically, as the control parameter R varies, we find that a new attractor arises at a critical value denoted as R_c. When $R < R_c$, the new attractor is still complex and is responsible for transients in many trajectories. We obtain numerically the exponent z which characterizes this transient. This paper is organized as follows: In Sec. 2, we discuss the behavior of the logistic map with constant control parameter, and use scaling arguments to obtain the exponents z and β characterizing the exchange of stability, tangent and period-doubling bifurcations. Section 3 treats the case of a periodic control parameter. We succeed in obtaining the fixed points and their stability. In addition, we describe the mechanism of the creation of the new attractor, the observed transient time, and we obtain the exponents z and β. Our conclusions are presented in the final section.

2. Constant Control Parameter

Let us consider the logistic map defined by Eq. (1) with $R_n = R$ ($\varepsilon = 0$). We are interested in the behavior of a typical dynamical quantity near a bifurcation. In particular, we shall investigate the variable defined by $Y = F^n(X) - X^*$, for the exchange of stability bifurcation, or by $Y = F^n(F^m(X)) - X^*$, for the period-doubling bifurcations ($m = 2, 4 \ldots$). In each case X^* stands for the asymptotic value of $F^m(X_n)$ (stable fixed point or periodic orbit) as $n \to \infty$ and at the bifurcation ($R = R_c$). Then Y will carry information of the approach or deviation from this typical value from both the dynamical and parametric points of view. Y, like any dynamical quantity, is a function of the number of iterations n (which, near a bifurcation, can be considered as a continuous variable) and of $\mu = |R - R_c|$. The dynamic scaling hypothesis [Hohenberg & Halperin, 1977] asserts that it should be a generalized homogeneous function:

$$Y(n, \mu) = lY(l^a n, l^b \mu)\,, \tag{3}$$

where l is a scaling factor. For $\mu = 0$, this implies that Y has an algebraic decay

$$Y(n, 0) \sim n^{-\beta}\,,$$

with $\beta = 1/a$. Figure 1 displays the numerical iteration of Eq. (1) for $R = 3$ (and $\varepsilon = 0$), where the first period-doubling bifurcation occurs. Observe the decay of $F^n(X)$ to $X^* = 2/3$ (the value of the fixed point) as a function of n. The upper curve is fitted with $Y = F^2(X) - 2/3$; the best fit gives $\beta = 0.493(3)$. When the lower curve is considered, the same value for β is obtained within uncertainty. For $\mu \neq 0$, the dynamical quantity has a decay characterized by a relaxation time τ. Near the bifurcation we expect that

$$\tau \sim \mu^{-z}\,.$$

Using the scaling relation [Eq. (3)] we find that $Y(n, \mu)$ goes as $n^{1/a}$ times a function of $n/\mu^{a/b}$, and so it follows that $z = -a/b$. The exponent z is called the dynamical critical exponent in the terminology of phase transitions [Hohenberg & Halperin, 1977]. In Fig. 2 we show how the relaxation time depends on μ for the exchange of stability bifurcation (at $R_c = 1$, when the fixed point $X = 0$ becomes unstable and so $X^* = 0$). The best fit gives $z = 0.9991(1)$.

Finally we consider Eq. (3) in the limit $n \to \infty$ in order to describe the divergence of dynamical quantities when approaching bifurcations in the orbit diagram. It is easy to obtain

$$Y(\infty, \mu) \sim \mu^{-1/b}\,.$$

Exponents a and b can be evaluated by standard renormalization group techniques. This method was applied to one-dimensional maps in order to describe the cascade of period-doubling bifurcation leading to the chaotic regime [Feigenbaum, 1978; Collet & Eckmann, 1980]. The method was later extended to describe intermittency [Hirsch *et al.*, 1982]. In this section, we apply the ideas used to characterize intermittency to describe relaxation at tangent bifurcations.

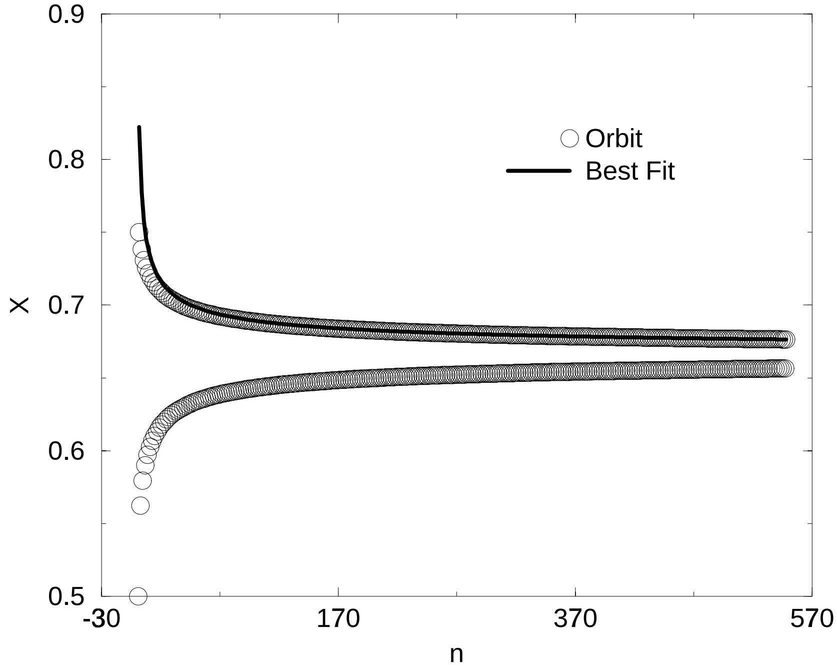

Fig. 1. Position X versus iteration number n for constant control parameter $R = 3$ and initial condition $X_0 = 1/2$. Each iteration alternates between the upper and lower curves while approaching the fixed point $X^* = 2/3$. Both curves decay as $n^{-\beta}$. A least squares fit furnishes $\beta = 0.493(3)$.

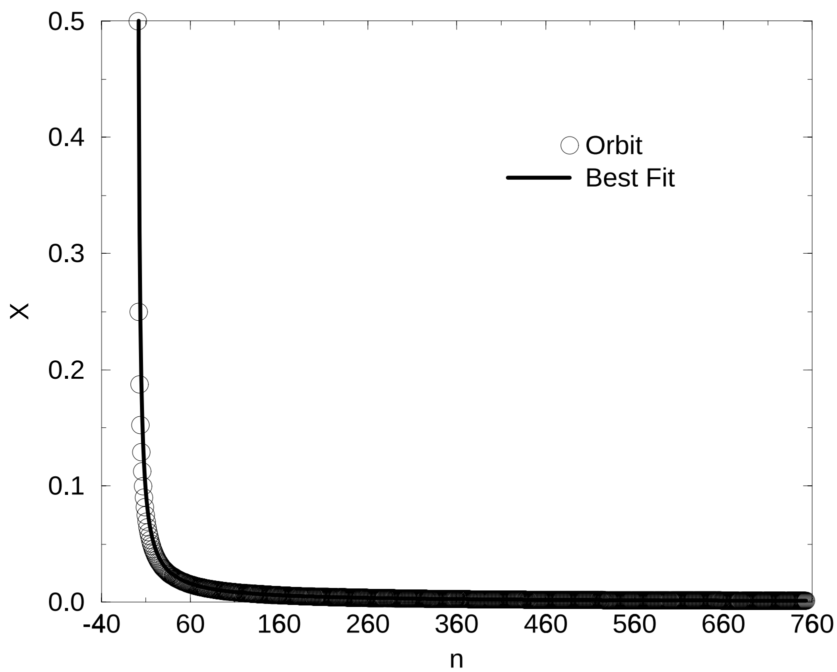

Fig. 2. Relaxation time versus μ for the exchange of stability bifurcation ($R_c = 1$) in the constant control parameter case. Here $\mu = R_c - R$. We obtain $z = 0.9991(1)$.

A tangent bifurcation occurs when the graph of the map intercepts the diagonal. When we are close to a tangent bifurcation, a channel between the graph of the map and the diagonal appears. If an orbit of the system enters such a channel it can spend a long time before leaving it; this time is identified as the relaxation time. The renormalization group approach states that near a tangent bifurcation, the logistic map behaves as universal function, $g(Y,\,\mu)$, which obeys the scaling relation

$$g(Y,\,\mu) = \alpha g\left(g\left(\frac{Y}{\alpha},\,\frac{\mu}{\delta}\right),\,\frac{\mu}{\delta}\right). \qquad (4)$$

The coefficients α and δ can be related to a and b by comparing the scaling relation above Eq. (3). α is identified as a *natural* scaling factor and thus $l \approx \alpha$. It follows that $l^b \approx 1/\delta$. Since the time scales n and n' defined by g and $g^{(2)}$ are related by $n' = 2n$, it also follows that $l^a n \approx 2n$. So we have $a = \ln(2)/\ln(\alpha)$ and $b = -\ln(\delta)/\ln(\alpha)$, implying that the exponents z and β are related to the renormalization group exponents by

$$z = \frac{\ln 2}{\ln \delta}\,,$$

$$\beta = \frac{\ln \alpha}{\ln 2}\,.$$

Assuming that $g(Y,\,\mu)$ has a power series expansion, we shall find α and δ by solving Eq. (4) recursively for each order of powers of Y and μ. The conditions $g(0,\,0) = 0$ and $(\partial g/\partial y)(0,\,0) = 1$ are appropriate for the tangent, exchange of stability and period-doubling bifurcation cases. Up to order 3, $g(Y,\,\mu)$ can be written as

$$g(Y,\,\mu) = Y + \sum_{j=2}^{3} c_{j0}Y^j + \sum_{k=1}^{3} c_{0k}\mu^k + \sum_{j=1,k=1}^{j+k\le 3} c_{jk}Y^j\mu^k\,.$$

Returning to the expansion in Eq. (4), we can obtain α and δ by considering terms in Y, μ, and in the product μY. By considering terms in Y alone, namely,

$$Y + c_{20}Y^2 + c_{30}Y^3 = Y + \frac{2c_{20}}{\alpha}\,Y^2 + \frac{2}{\alpha^2}\,(c_{30} + c_{20}^2)Y^3\,,$$

we can determine α. If $c_{20} \neq 0$ then $\alpha = 2$. This situation is found in the tangent and exchange of stability bifurcation cases. If $c_{20} = 0$ and $c_{30} \neq 0$, we have $\alpha = \sqrt{2}$. This corresponds to a period-doubling bifurcation. In fact, this series can be summed in all order in Y [Hirsch *et al.*, 1982], yielding a function depending only on a single free constant parameter (c_{20} or c_{30}).

Concerning the terms in μ only, we have that

$$\sum_{j=1}^{3} c_{0j}\mu^j = \frac{2\alpha c_{01}}{\delta}\,\mu + \frac{\alpha}{\delta^2}\,(2c_{02} + c_{01}c_{11} + c_{20}c_{01}^2)\mu^2 + \frac{\alpha}{\delta^3}\,(2c_{03} + c_{11}c_{02} + c_{12}c_{01} + 2c_{20}c_{01}c_{02} + c_{21}c_{01}^2 + c_{30}c_{01}^3)\mu^3\,.$$

When $c_{01} \neq 0$, it follows that $\delta = 2\alpha$. If $c_{01} = 0$ and $c_{02} \neq 0$ we have $\delta = \sqrt{2\alpha}$ and so on. However, in order to determine δ, we must consider simultaneously the series with the cross terms, namely

$$\begin{aligned} c_{11}Y\mu + c_{12}Y\mu^2 + c_{21}Y^2\mu &= \frac{2}{\delta}\,(c_{11} + c_{20}c_{01})Y\mu + \frac{2}{\delta^2}\left(c_{12} + c_{11}^2 + c_{20}c_{02} + c_{20}c_{01}c_{11} + c_{21}c_{01} + \frac{3}{2}\,c_{03}c_{01}^2\right)Y\mu^2 \\ &\quad + \frac{2}{\delta\alpha}\left(c_{21} + \frac{3}{2}\,c_{20}c_{11} + \frac{3}{2}\,c_{30}c_{01} + c_{20}^2c_{01}\right)Y^2\mu\,, \end{aligned}$$

and check for consistency.

Since c_{20} and c_{01} are different from zero at tangent bifurcations, we obtain $\alpha = 2$ and $\delta = 4$. This implies that $\beta = 1$ and $z = 1/2$. In the exchange of stability bifurcation, which occurs for $R_c = 1$ and $X^* = 0$ in the logistic map, we have no terms with μ alone. The exponents and the coefficients are determined by $c_{20} \neq 0$ and $c_{11} \neq 0$. In this case, we have that $\alpha = 2$ and $\delta = 2$, implying that $\beta = 1$ and $z = 1$. The period-doubling bifurcation is characterized by $c_{20} = 0$ and $c_{30} \neq 0$. Moreover, the coefficients c_{0j} are all zero and $c_{11} \neq 0$. Thus we have $\alpha = \sqrt{2}$ and $\delta = 2$, implying $\beta = 1/2$ and $z = 1$.

Considering terms up to third order in Y and μ, we see that we can have only two values for the exponent β, namely 1 or 1/2. On the other hand, several values of z are possible: they are $z = 1/2$, 2/3, 1, 3/4 and 2.

It is worth mentioning that the results obtained above hold for all one-dimensional maps in the same universality class and are not restricted to the logistic map.

3. Periodic Control Parameter

Let us consider now the logistic map with a control parameter of period 2. This problem can be modeled by Eqs. (1) and (2) with $f_n = f(n) = \cos(n\pi)$.

Since R_n oscillates between two values, $R(1+\varepsilon)$ and $R(1-\varepsilon)$, the fixed points are also periodic in time (n). This means that $f_n(X_a) = X_b$ and $f_{n+1}(X_b) = X_a$. We prefer to call this situation a time dependent, in fact a periodic, fixed point rather than a two-cycle, as almost all initial conditions will asymptotically oscillate between X_a and X_b. So we have a time-dependent fixed point. To clarify the difference between a periodic fixed point and a two-cycle, suppose that $R_n = R$ and we are given many different initial conditions. If the system evolves to two-cycle (X_a, X_b), each X_i ($i = a, b$) has its own basin of attraction. Thus at any given n some initial conditions will be close to X_a and almost all the other will be close to X_b. At iteration $n+1$ they exchange values. On the other hand, when the control parameter has period two, almost all initial conditions have the asymptotic value X_a for a given n and the value X_b for $n+1$. Our case is then better described by a time-dependent fixed point of period two.

The fixed points are obtained by solving the equation $X^*_{n+2} = X^*_n$ where iteration is defined by Eqs. (1) and (2). This can be easily accomplished using a symbolic programming tool such as Maple. We find three nonzero solutions for n even (odd). At least one of them is real and corresponds to a fixed point $\mathbf{X}^*_1 = (X_a, X_b)$. If we fix ε and increase R, this fixed point becomes unstable and a two-cycle (of period two) becomes stable. So, as R is varied we have the usual sequence of period doublings, until chaotic behavior is reached.

The other two roots of the fixed point equation change from complex to real upon increasing R. The value of the control parameter where complex fixed points become real is called R_c. One of these new fixed points is unstable and the other is stable. This new stable fixed point $\mathbf{X}^*_2$ coexists with the original real fixed point or with one of its stable descendent m-cycle. It has its own basin of attraction. In fact, some of the initial conditions belong to the basin of attraction of $\mathbf{X}^*_2$, while others belong to the basin of the stable m-cycle ($m = 1, 2, 3, 4\ldots$), which appears in the sequence of bifurcations originated from $\mathbf{X}^*_1$. The new fixed point, $\mathbf{X}^*_2$, gives rise to a second sequence of bifurcations as R is varied.

The picture described above depends on ε. For example, for $\varepsilon = 0.01$ we have $R_c \sim 3.144390069$. For $R < R_c$ we have only a fixed point of period two, $\mathbf{X}^*_1$, that attracts almost all initial conditions. When $R > R_c$ we have two stable fixed points of period two, $\mathbf{X}^*_1$ and $\mathbf{X}^*_2$, each one with its own basin of attraction. The sudden birth of a new basin of attraction can be seen in the orbit diagram as a discontinuity. In Fig. 3 we show the orbit diagram for this value of ε and two different initial conditions:

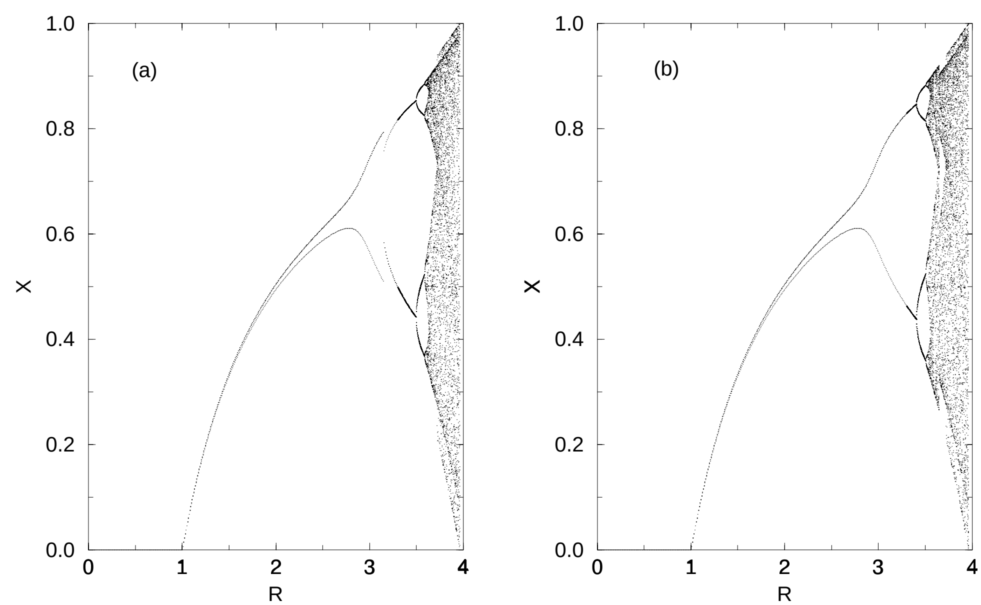

Fig. 3. Orbit diagram for periodic control parameter $R_n = R(1 + \varepsilon \cos(n\pi))$ with $\varepsilon = 0.01$. (a) The initial condition is $X_0 = 1/2$. For $R < R_c = 3.1443900\ldots$, X_0 is attracted by the real periodic fixed point; for $R = R_c$, we have a new real periodic fixed point responsible for a second basin of attraction. The discontinuity at R_c is due to the fact that X_0 is now attracted to the new fixed point. (b) The initial condition is $X_0 = 0.2$.

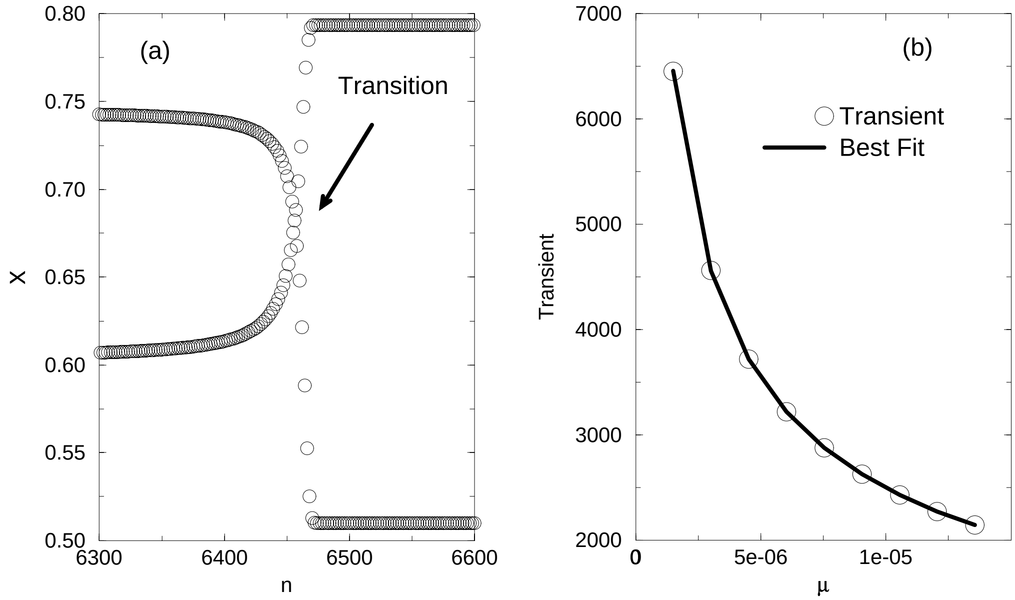

Fig. 4. The control parameter is given by $R_n = R(1 + \varepsilon \cos(n\pi))$ with $\varepsilon = 0.01$. (a) Orbit of $X_0 = 1/2$ when $R < R_c = 3.1443900\ldots$. In this situation the new attractor is complex and responsible for a transient. The orbit spends a long time near the real part of the complex attractor ($n < 6450$) before reaching the real periodic fixed point. The arrow shows the transition defining the transient time. (b) Plot of transient time versus μ. A least squares fitting furnishes $z = 0.501(1)$.

(a) $X_0 = 0.5$ and (b) $X_0 = 0.2$. The discontinuity at R_c in Fig. 3(a) means that $X_0 = 0.5$ has changed its basin of attraction. This does not occur for $X_0 = 0.2$ [see Fig. 3(b)].

This picture describes the mechanism by which new basins of attraction appear. The transition from one basin to another one may be characterized by a transient time. For $R < R_c$, an initial condition X_0 is attracted asymptotically to the fixed point $\mathbf{X}_1^*$, but before *reaching* this fixed point the system may spend a long time near the real part of the (still) complex fixed point $\mathbf{X}_2^*$. This time is the transient time τ. As R approaches R_c, τ grows, and for $R = R_c$ it diverges, meaning that now the system spends all the time in the neighborhood of the new real fixed point. The divergence of the transient time near R_c is given by $\tau \sim \mu^{-z}$. Figure 4(a) shows the evolution starting from $X_0 = 0.5$. Observe that the transition from one fixed point to the other is very sharp. It is then easy to evaluate the transient time. Figure 4(b) shows the transient time for different values of $\mu = R_c - R$. The exponent $z = 0.501(1)$ was obtained by a least squares fitting; the quality of the fit is very good.

A similar situation may occur when a complex fixed point becomes real and stable when a m-cycle is already stable. This is the case for $\varepsilon = 0.038$. The fixed point $\mathbf{X}_2^*$ becomes real and stable at $R = R_c \sim 3.372405139$. When $R < R_c$, $\mathbf{X}_2^*$ is complex and a two-cycle of period two attracts all the initial conditions. The transient time, in which the system stays close to the real part of $\mathbf{X}_2^*$ is characterized by the exponent $z = 0.506(2)$.

For $\varepsilon = 0.053$ and $R < R_c \sim 3.477265810$ we have a chaotic attractor and a transient occurs between it and the real part of the complex fixed point. This is shown in Fig. 5(a). The transient times for several values $\mu = R_c - R$ are shown in Fig. 5(b). A least squares fit furnishes $z = 0.502(2)$.

The transient described above may be compared to intermittency. In intermittent behavior we have coexistence between an established chaotic regime and a m-cycle which would be the attractor for a slight change in the control parameter. There is a channel between the graph of the map $F^{(m)}$ and the straight line $y = x$. As an initial condition is iterated it can enter this channel and thus have a cycle-like behavior. When it leaves the channel we have a so-called burst of chaotic behavior. It turns out that there is a reinjection mechanism so the system enters and leaves the channel frequently. The characteristic time τ is defined as the average time of bursts and is characterized by $z = 0.5$. In our case, we also have a channel related to a tangent bifurcation, but there is no reinjection mechanism. In fact, the system enters the channel only once

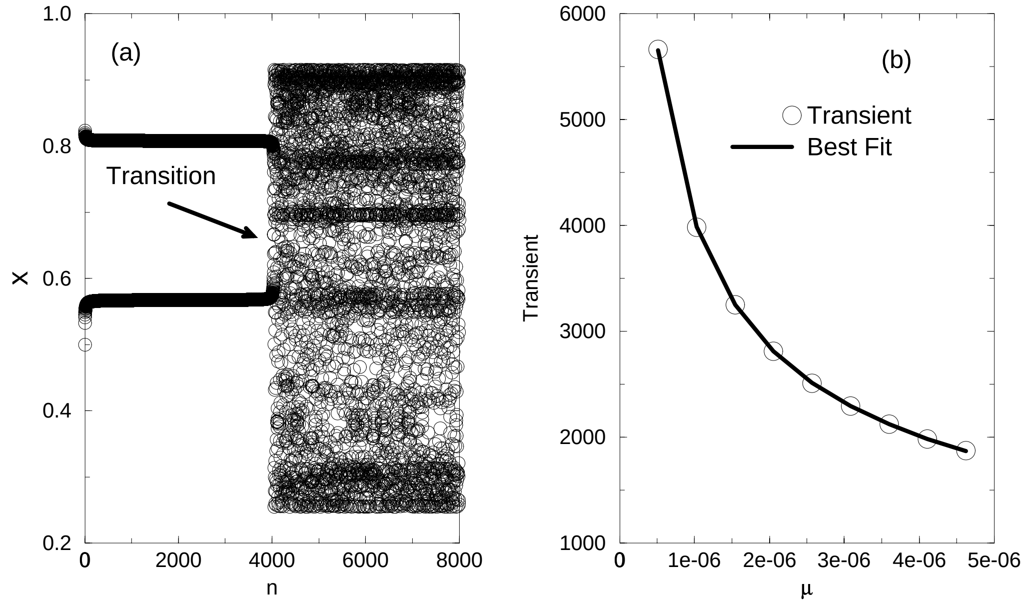

Fig. 5. (a) The orbit of the initial condition $X_0 = 1/2$ for $R_n = R(1+\varepsilon\cos(n\pi))$ with $R < R_c = 3.47726581\ldots$ and $\varepsilon = 0.053$. There are a complex periodic fixed point and a real chaotic attractor. A sharp transition marks the transient time. (b) Plot of the transient time versus $\mu = R - R_c$. A least squares fit gives us $z = 0.502(2)$.

and spends a long time there, which also seems to be characterized by $z = 0.5$. Finally it passes to the chaotic attractor.

Similar situations are observed when the control parameter R_n has period three. Using, for example, $R_n = R(1 + \varepsilon\cos(2/3n\pi)$, we have three possible values for R_n, namely $R_a = R(1 - \varepsilon/2)$, $R_b = R(1-\varepsilon/2)$ and $R_c = R(1+\varepsilon)$. In order to obtain the fixed points, we need to solve the equation

$$X^*_{n+3} = X^*_n\,,$$

which give us seven nonzero roots for each value of $n = 1, 2, 3$. For $\varepsilon = 0.01$, for example, one of the seven roots is at first real and the others are complex. These complex solutions become real in pairs at specific values of R.

This occurs for the first time when $R_1 = 3.783544316\ldots$, $R_2 = 3.850718107\ldots$ and $R_3 = 3.851414845\ldots$. We again observe similar transients and a similar mechanism of creation of the new attractor. The exponents z can be obtained numerically; we have found values close to 0.5 for all transients.

4. Final Remarks

We studied the logistic map in the case of: (a) constant control parameter and (b) periodic control parameter. In the first case we used arguments and techniques of the renormalization group in order to obtain the dynamical exponents z and β, which characterize, respectively, the relaxation time near a bifurcation and the time decay at the bifurcation. We obtained the following results: (i) $z = 1/2$ and $\beta = 1$ for tangent bifurcations; (ii) $z = 1$ and $\beta = 1$ for period-doubling bifurcations; (iii) $z = 1$ and $\beta = 1/2$ for the exchange of stability bifurcation.

In the second case we introduced the concept of the periodic fixed point, and succeeded in identifying and analyzing it. We described how a complex solution becomes real at $R = R_c$, implying the creation of a new attractor. For $R < R_c$, we obtained the exponent z related to a transient time. For $R_n = R(1 + \varepsilon\cos(n\pi))$ we found that R_c depends on ε. Also, depending on ε, the new attractor may arise when the old attractor was a periodic fixed point, a periodic m-cycle ($m = 2, 4\ldots$), or chaotic. For each situation, the exponent z was found to be approximately 0.5. The same results were observed for R_n with period three, i.e. $R_n = R(1 + \varepsilon\cos(2/3n\pi))$.

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