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A STUDENT'S CONSTRUCTION OF TRANSFORMATIONS OF FUNCTIONS IN A MULTIPLE REPRESENTATIONAL ENVIRONMENT[1]

ABSTRACT. This paper reports on a case study of a 16-year-old student working on transformations of functions in a computer-based, multi-representational environment. The didactic approach to reflections, translations and stretches began with visualization exercises, and then was extended to investigate the implications of visual changes in data points, and subsequently, in algebraic symbolism. A detailed analysis of the student's work during the transition from the use of visualization and analysis of discrete points to the use of algebraic symbolism is presented. Two new semiotic forms are introduced as an alternative kind of algebraic symbolism and as a means to facilitate the transition to $f(x)$ notation: covariational equations and the horseshoe display for transformations. The implications of this case for the redesign and modification of the software are discussed.

AN APPROACH USING GRAPHICAL VISUALIZATION

In light of the increasing availability of new media such as computers, as well as underlying historical and philosophical issues that have been raised (Borba, 1993; Confrey 1993a; Denis and Confrey, 1995), the dominance of symbolic algebra over other representations in the teaching of mathematics needs to be questioned. To this end, we postulated a general approach to teaching transformations that begins with graphical visualization. It is followed by generation of tabular data from the graphs and investigation of how this data changes under different transformations, working towards the development of traditional algebraic symbolism (in $f(x)$ form) as a semiotic code for describing the regularities found in the other two representations. This study examines the viability of such an approach within a particular computer environment and describes emergent semiotic forms that aided the student in accomplishing the tasks and aided us in interpreting the student's actions. Our focus will be on the transition from the use of visualization and discrete points to the use of algebraic symbolism.

Traditionally, transformations of functions have been taught with a strong emphasis on algebraic symbolism and in relative isolation from the visual transformational topics in geometry. The typical approach to transformation in most conventional textbooks varies the coefficients of a

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function and examines the resulting changes in the graph. For instance, the question is asked: if $y = x^2 + 5$ is changed to $y = 2x^2 + 5$, how does the graph of the transformed function change? Confrey (1994a) calls this a template approach and documents student difficulties including a tendency to memorize rules without understanding their genesis and failure to make the subtle distinctions among different symbolic forms. Other studies in which such an approach is used document students' difficulties in generalizing the patterns, including problems with moving among different families of functions, and problems of visual screen constraints and scaling issues (Eisenberg and Dreyfus, 1991; Goldenberg et al., 1988; Goldenberg and Kliman, 1990). These studies relied on technologies in which changes to the graphs were indirect, through the algebraic symbolism or through simple command codes. In this study, however, the transformations were made more directly through mouse actions; consequently the findings differ somewhat.

For this research, we elaborated an approach which inverted the order of the traditional approach to transformations of functions (Borba, 1993; Confrey, 1994a). The approach starts with visualization of graphs, then focuses on the relationship between graphs and tabular values and, finally, focuses on the relationship between graphs and algebraic representations[2]. We believe that the emphasis on visualization, in addition to restoring the historic emphasis on visual aspects in the study of transformations, allows students to move easily into the algebraic symbolism while maintaining some visual meaning for the symbolism. This report concentrates on the final transition to algebraic symbolism and on how the previous experiences in visualization and the manipulation of data points facilitated that transition.

THE STUDY

Teaching experiments (Cobb and Steffe, 1983) were conducted to build models of how students think about transformations of functions. In this methodology, which is a variation of clinical interviews, students and researchers are expected to learn from each other as students proceed through a series of tasks designed by the interviewer. The role of the interaction between interviewer and interviewee is acknowledged as an intervention, hence the label, teaching experiment. It is further assumed that, "when students engage with a genuine problem, struggle to resolve it, and reflect on those actions, learning of a very profound sort has occurred." (Confrey, 1991b, p. 128). One of the participating students, Ron, was interviewed during eight approximately two-hour sessions over a five-week

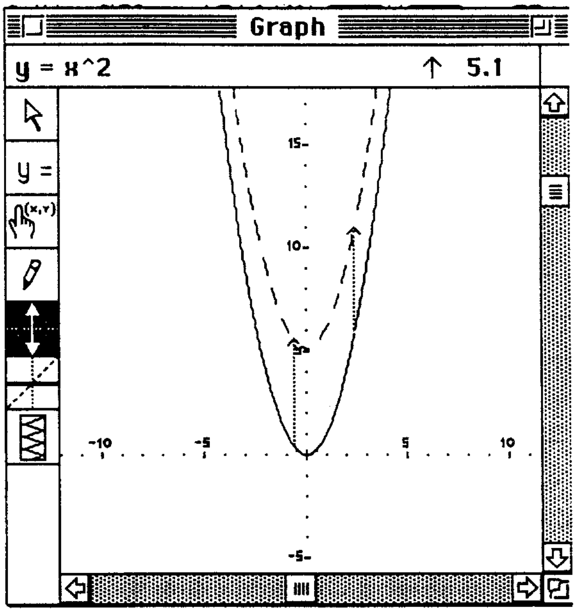

Figure 1. A vertical translation by 5.1 made through "direct" actions on the graph. Notice that, since the command 'hide transformations" is on, the equation of the transformed parabola does not show on the upper left corner. The transformation could have been done through the keyboard. In this case, the magnitude of the translation would show in the upper right corner.

period. These were video and audiotaped, and transcriptions prepared. He had at his disposal paper and pencil and an Apple Macintosh Computer (IIsi) with the software, Function Probe.

Ron was sixteen years old at the time the interviews took place. He was considered a good student by his teachers. He had completed Course 3, part of an integrated mathematics program mandated by the state, which included an introduction to functions but did not include transformations of functions as a topic. He was not taking any further mathematics courses at the time the interviews began. Halfway through the teaching experiment, he started an independent study with a teacher from his school, The Alternative Community School in Ithaca, NY. Ron volunteered for the study in response to an announcement, and he was paid for his participation[3].

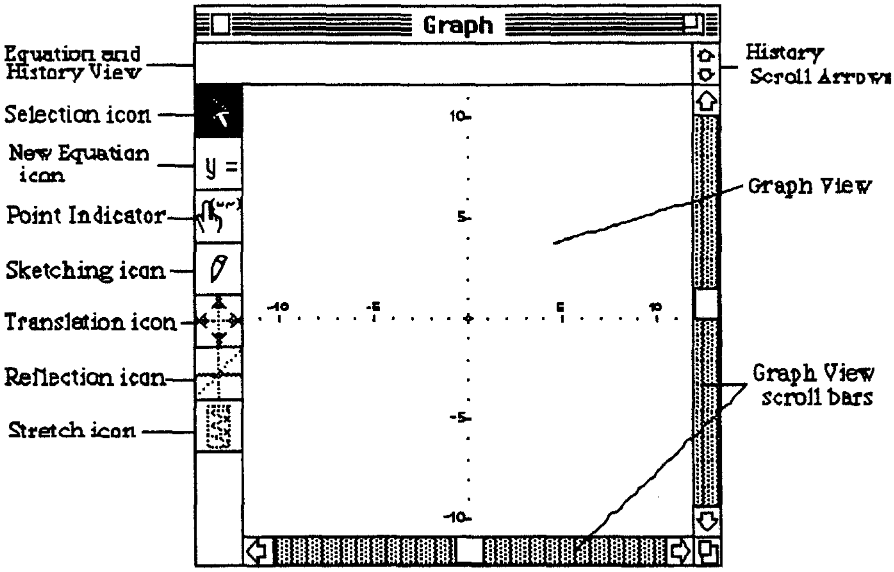

Figure 2. An overview of the graph window.

A sequence of tasks was developed based on the graphical visualization approach. Ron was introduced to the features of Function Probe during three tutorials conducted in the course of the teaching experiment. In the tutorials, cubic functions were used as a means to demonstrate the capabilities of the software. In the main part of the experiment, three other functional families were used: absolute value, quadratic and step functions. For each of the families of functions used in the experiment, three main problems were repeated. First, Ron was asked to match different pairs of graphs using the graph window icons and mouse actions. This task emphasized visualization and mouse actions as a way of investigating transformations of functions. Secondly, Ron was asked to transform the basic graph shape for the functions $y = |x|$, $y = x^2$ and $y = [x]$ in various ways and to predict the effect of the transformations on tabular values which he selected using the sampling tool on the software. Finally, Ron was asked to study the relationship between changes in different graphs and changes in coefficients of their algebraic equations.

In this study, vertical and horizontal translations, reflections around vertical and horizontal lines, and vertical and horizontal stretches were investigated.

THE SOFTWARE

An alternative to overemphasizing algebraic symbolism lies in the use of multi-representational software such as Function Probe© (Confrey, 1991a). The software design is a result of years of teaching and research conducted by the Mathematics Education Research Group at Cornell University[4]. The design of the software is therefore heavily influenced by students' reasoning. We see this as an ongoing process: the design of Function Probe (**FP**) was based on research on students' reasoning, and new research continues to inform the redesign of the software.

FP is a multi-representational software for Apple Macintosh computers. At the time the teaching experiment was undertaken, the software had three windows: graph, table and calculator window. Since the calculator was used minimally in this experiment, its features will not be discussed.

The graph window allows the user to graph functions in a number of different ways: 1) inputting algebraic symbolism for the function; 2) plotting point by point in the (x, y) format; 3) sketching using the mouse as if it were a pen; 4) calling a "calculator button"; or 5) "receiving" ordered pairs from a table. In addition, FP has special features for transformations of functions. The graph of a function can be transformed by mouse actions using the bottom three icons on the left hand side of Fig. 1. Alternatively, the value for the transformation can be typed directly in the upper right hand corner of the graph window. The icon palette in the graph window of Function Probe allows for actions – such as horizontal and vertical translations, horizontal and vertical stretches and reflections across vertical and horizontal lines – to be made "directly" to the graph. The algebraic symbols resulting from the "mouse driven" transformations can be either hidden or displayed using a menu option[5].

Scaling in the graph window is not automatic. Re-scaling in the graph window requires the use of the "rescale box", so that students have to actively identify the intervals in the "x" and "y" axes that they want to be displayed as well as the size of the unit they wish to be marked off in tics. Scales can be saved for re-use. A history of all equations drawn, sketched, deleted or transformed is kept in the history window.

In addition to the translation, stretch, reflection and new equation icons, the icon palette also has the sketch and the point indicator icon which are relevant to this study (see Fig. 2). The former allows students to make a free hand drawing of a graph, while the latter allows the identification of the coordinates of a point in the Cartesian plane. The sketch icon was used frequently as a vehicle for obtaining the student"s predictions The point indicator can be used as a measuring tool. Another feature of FP, which was fundamental for this study, was the sampling tool that is accessible through the graph menu. The sampling tool allows a finite subset of the function to be selected and "sent" to the table[6]. Sampling proved to be a significant tool that allowed the student to relate transformations in the graph to numerical changes in the ordered pairs of a function.

The table window facilitates entry of patterns of data in tables – a task that would be otherwise tedious. "x" and "y" columns can be generated independently using the "fill" command. For example, this command allows the user to fill one column in an arithmetic progression (e.g.. the "x" column starting at zero, adding one until 50) and another column in a geometric progression (e.g. the "y" column in a geometric progression starting at 100 with rate "1/2" until it reaches the fiftieth place). If one column has already been generated, a second column can be generated as a function which uses the variable of the first column as independent variable. Regardless of how a pair of columns is generated, they can be "sent" to the graph window in the (x, y) form where the function, having a discrete domain and range, is plotted in the Cartesian plane.

A CASE STUDY

To review, the approach for teaching transformation of functions included three basic steps: a visual one based on actions on the icons of the graph window of FP; a visual-discrete one in which a student was invited to connect his/her visual experiences with the discrete world of points in the Cartesian plane and coordinates displayed in the table window; and the final one in which a student was expected to connect visualization and the patterns in the numeric data with algebraic symbolism eventually notated in the form $y = ...$

Ron proceeded through the first two tasks of the teaching experiment quickly and with few obstacles (Borba, 1993) . For instance, when asked to match the graphs of two different parabolas using the icons of Function Probe, he did not hesitate to reflect one of them if necessary, translate the reflected graph until the vertexes of the two coincided, and then horizontally stretch the graph until the curves coincided.

In the second part of the teaching experiment, Ron showed that he could predict and explain the relationship between the visual and the numeric values of the points easily and swiftly. For instance, when reflecting a graph about a line of reflection that was parallel to the axes, he predicted the impact on the values of the discrete points rapidly, stating:

> Let's say the reflection line is at 10... Then at this point at 0 would reflect up to 20. A point at 10 would stay right where it is. A point at 5 would reflect up to 15. And you can ... I guess the formula for that would be you take the y-value that you have, you subtract 10, multiply by negative 1, and then you add your 10 back in.

Next, when he was asked to create symbolic descriptions of the transformations, he generated equations to describe a transformation in the table representation in terms of the old x, the new x, the old y and the new y. For example, just after stating the above explanation, he coded the reflection around $y = 10$ as $(y[-10]) \bullet -1[+10]$, which could be interpreted as $y' = (y - 10) * (-1) + 10$. Borba (1993, 1994a) has labeled equations like these "covariational equations" and offered the formalization of these symbols as: $x' = x$ and $y' = (y - 10) * (-1) + 10$ where (x', y') are the coordinates of the translated set of points and (x, y) are the original set of points. These covariational equations express a type of algebra which is in contrast to the $f(x)$ form, mapping x to $f(x)$.

To complete the third step of the visualization approach, Ron was asked to study the relationship between changes in different graphs and changes in coefficients of their algebraic equations. Specifically, he was asked to choose any form for representing a quadratic algebraically (these forms had been used in previous coursework) and try to describe how the coefficients were related to translations, stretches and reflections.

If Ron had not been familiar with a "general form" for quadratics, the interviewer was prepared to suggest working with the format $y = Af(Bx + C) + D$ applied to quadratics, hence $y = A(Bx + C)^2 + D$. This is because this form is 1) relatively straight-forward to relate variable by variable to the desired transformations, and 2) it is closely related to the underlying form designed into the software to create the algebraic representation. However, Ron decided to first investigate the quadratic function using the formula $y = ax^2 + bx + c$ which was more familiar to him.

The Problematic

Tackling this problem, Ron rapidly found, using the icons and the rescale facilities of FP, that a Horizontal Translation (HT) by 5 to the right would imply a change in "c" from 5 to 15 in the equation $y = x^2 + 3x + 5$. He moved the mouse to the right, observed the graph moving 5 units to the right, noted that the graph crossed the y-axis at $y = 15$, and therefore concluded that $c = 15$. He then decided to check his finding using paper and pencil. He claimed:

... I know an easy way to find out what c will be ... Just substitute ... do the formula that I did [the covariational equation discussed before]. y ... so x, x^2+ ... I think this will work ... $3x+5$, and the thing ... transformation was to the right by ... It's $(x+5)^2+3(x+5)+5$.

It appears that Ron built on his previous analysis of covariational equations to propose that, in a HT where the new x is increased by 5 ($x' = x+5$), the new equation can be arrived at by substituting $x+5$ for x.

Ron's notation for this was "$(x, x^2+3x+5) \rightarrow (x+5)^2+3(x+5)+5$". Next he simplified the second expression to get "$(x+5, x^2+13x+45)$". Now Ron was faced with a discrepancy between the results he had found using the icons (that a HT by 5 to the right would make "c" change to 15) and those he got with his algebraic algorithm (that HT by 5 to the right would make "c" change to 45).

Until this point, the "hide transformations" feature of FP had been on, so that the equations of the transformed functions were not being shown. The interviewer now turned it off and FP displayed "$y = (x-5)^2+3(x-5)+5$" for the function which had been translated to the right by 5. Ron's first reaction indicated that he had not expected these results: "I went the wrong way", referring to his use of $(x+5)$ instead of $(x-5)$. Ron now had a discrepancy in his results: using the icon, he found $c = 15$; using covariational equations he found $c = 45$.

A Graphical and Tabular Approach

Ron was convinced that his calculation based on the covariational equations, which led him to find $c = 45$, was correct, but he was also convinced that what he saw in Fig. 3 was right. His first attempt to resolve the discrepancy between the two results was an appeal to a visual argument. He pointed to the graphs of $y = x^2+3x+5$ and $y = (x-5)^2+3(x-5)+5$ (see graphs B and C in Fig. 3), and argued that the latter function would have 5 subtracted from its x-coordinates so that it would have the same y-value as the former function. However, when he was asked by the interviewer whether he was "happy with this explanation", he answered that "my logic isn't finished ... because this is $x+5$". By "this", it appeared that Ron was referring to the set of new x values displayed both in the graph and table window.

A tension between two kinds of reasoning seemed to underlie Ron's previous argument: when he saw the transformation as dynamic, as a movement to the right, he felt there should be a plus sign in the algebraic expression of the transformed function just as there had been in the covariational equations. Instead he found a minus sign in the Function Probe algebraic display. On the other hand, when Ron saw the transformation as a static comparison of graphs B and C, as something which had already

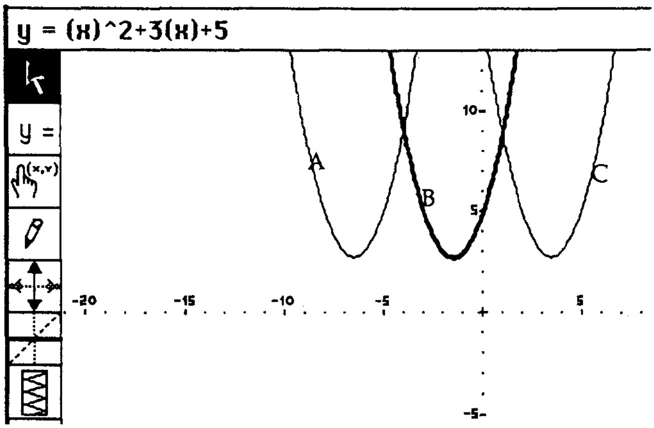

Figure 3. From left to right, the graphs of $y = (x+5)^2 + 3(x+5) + 5$, $y = x^2 + 3x + 5$ and $y = (x-5)^2 + 3(x-5) + 5$.

occurred, he found an explanation for the minus sign, but it was still not clear to him why there was not a plus sign as suggested by his investigations on the table representation. This explains why he remained puzzled in spite of having found an explanation for the discrepancy. The fact that, in his covariational equations, a plus sign was associated with horizontal translations to the right and vertical translations up, while minus signs were associated with the directions left and down, continued to produce an unresolved conflict for Ron as he tried to connect covariational equations, transformations on the graphs and $y = f(x)$ forms.

A Transition to f(x) Notation

In this episode, Ron tried to connect his covariational equations and $f(x)$ notation. By coordinating the visual, tabular and symbolic forms, Ron reached an understanding of the equation displayed on FP.

First, Ron represented the translation to the right as $(x+5, f(x))$. This was a critical step for Ron, because it created a bridge between covariational form and $f(x)$ form. One interpretation was that Ron considered the original equation in the form, $(x, f(x))$ or in this case, (x, x^2+3x+5). His movement to the right was coded covariationally as "the new x being the old $x+5$", thus he wrote $(x+5, f(x))$ combining the two notational forms. Possibly his reasoning was that the $x+5$ coded the movement to the right, while the $f(x)$ notation assured that the y values remained as they were before. Because the interviewer did not probe further, this interpretation cannot be further confirmed.

His dilemma now was that he recognized that the software would not accept this notational form and thus, he was still unable to confirm whether this conjecture would resolve his problematic. Thus, he strove to find a way in which he could represent this new form, $(x+5, f(x))$ in the general form $(a, f(a))$ where "a" is an isolated variable. He proposed:

It's like this... you translate by 5 and you get $(x+5, f(x))$.... Then you subtract 5, $x+5-5$, so it's $x-5$... will give you $(x, f(x-5)$ there ... [because] you want x to be the same, so you do $(x, f(x-5))$.[7]

One way to interpret Ron's reasoning is as a change of variables. In this sense, $(x+5, x^2+3x+5)$ *(i)* can be changed into the form $y = f(x)$ if we make $x' = x+5$ *(ii)* in equation *(i)* and take into account that $y = x^2+3x+5$ *(iii)*. From *(ii)* we derive that $x = x'-5$ and, substituting in *(iii)*, we have $y = (x'-5)^2+3(x'-5)+5$ which is equivalent to $(x', f(x'-5))$ as proposed by Ron.[8]

The Creation of an Integrating Metaphor

The solutions described in the previous two episodes resolved the issue of explaining the minus sign and the discrepancy between $c = 15$ and $c = 45$ to the satisfaction of the interviewer, but as it turned out, not yet completely to Ron's satisfaction. The interviewer then directed Ron to investigate the relationship between a numeric change in the coefficient of b of the algebraic expression of a function $(y = ax^2 + bx + c)$ and its implications for the type, magnitude and direction of a transformation.

Ron could not connect "b" with any of the studied transformations (translations, reflections and stretches) and was about to give up when the interviewer suggested that he try to explain B in the format $y = A(Bx+C)^2 + D$. Ron accepted the suggestion, but he soon became interested in the issue of the minus sign again, now in this revised form. This led Ron to investigate the situation in which $A = B = 1, D = 0$ and to vary solely C, thus focusing on the family of functions $y = (x+C)^2$:

Wait, wait, I just want to change it back to $+5^2[(x+5)^2]$. That's all. So now if you imagine this x-axis shifting over completely by 5. Shifted +5, okay? ... Now, and change all the values back to what they should be okay? And then over here at -5 is going to be where the y-axis is going to be -5 is going to be, see what I'm trying to say?

Ron seemed to be building an argument in which he was seeing a HT as a "function" of moving the x-axis sideways in relation to its intersection with the y-axis instead of seeing the translation as a shift of the *graph*[9]. As the interviewer realized the nature of Ron's argument, he asked Ron if that could explain why, when C had a "+5 change", the graph would move to the left. Ron's response was quite telling:

> Um... yes. If you do... if you think about it right. Let me think about it. Yeah. See if you pull this [the x-axis] to the right, -5 is getting to end up at the zero, and the distance will be 0 at -5, see?

Ron's articulation suggests that he was understanding "+5" as a movement of the x-axis to the right which would shift -5 to the origin ("-5 is getting to end up at the zero") and resulting in having the vertex of the parabola at -5 ("the distance will be 0 at -5"). He goes on to claim that the x-axis would be pulled back to the right, because "[b]y being a function, it pushes it all back the way it was supposed to be... so then this whole thing slides back and then it's plotted where it is now...".

To interpret the meaning of these claims, we refer back to the earlier discussion of the meaning of the notation, $(x + 5, f(x))$. For Ron, this notation seems to mean that a function is composed of two sets of points, and that this notation signals a change in the x-axis to the right by 5 *while the y values, or $f(x)$ conserves its shape.* Thus, we see Ron treating the $f(x)$ as a separate object to be moved about on the plane as necessitated by changes in the x. However, Ron seems in his second utterance to be thinking that the axes must be returned to their initial positions, hence his claim; "it pushes it all back the way it was supposed to be, ... so then this whole thing slides back and then its plotted where it is now." This requirement to return the axes to their initial position may be perceptually motivated in order to explain the picture as it is produced in this software when the axes appear fixed and the curve is translated; or it may be a more global claim about the permanence of the origin in any medium.

A way to interpret Ron's reasoning is to imagine a metaphor of "double rubber sheets" proposed by (Borba, 1993) as a refinement of the "rubber sheet metaphor" (Goldenberg and Kliman, 1990). In the double rubber sheet metaphor, there is a back rubber sheet with the Cartesian axes and a front rubber sheet with the curve on it. Previously, the authors had chosen to think of transformations as impacting on the front rubber sheet while the back rubber sheet remained permanent. Ron, however, was thinking of the horizontal translation as being an action on the back rubber sheet (x-axis), followed by an adjustment which would push both rubber sheets, so that the origin would be back to its original position.

Notice that Ron's approach solves the problem of having a discrepancy between the sign in a horizontal translation and the direction of the movement. The x-axis moves in the direction of the sign of the numeric value, and thus Ron has made a "non-inverted" relationship between the change in the C coefficient and the action that is carried out on the graph.

An extension of this approach to changes in the coefficients corresponding to stretches and contractions poses an additional challenge for Ron. Recall that when there is an increase in B of magnitude 2, for instance, the graph shape contracts to one half its previous size, whereas an increase in A of magnitude 2 results in an expansion of the graph to twice its previous size. Thus, the analogous problem of "inversion" reappears in stretching and contracting. Can Ron's approach resolve this problem, and how does Ron go about investigating that question?

Ron continues to refine and explain his approach to his problematic:

And then you have a function. This pencil, okay? This is $y = x$ right here. Plop. And if you don't do anything, it just plots right on the paper, it sticks there, and that's what you get. Now when you say $y = x + 5$, shifts the graph over, plops it down, or shifts the graph over, plops it down and moves it back where it's supposed to be, okay? So now the functions move to the left just like that ... And no, so if you do $y = 2x$, then it says, take the whole graph paper and stretch it out, plop the x down, plop this – would that stretched piece of graph, will that stretched piece of paper, put this on there, then push it back to where it's supposed to be and this whole thing will go like this.

At this time, the interviewer could see what Ron had in mind and asked him if thinking of this as a "double rubber sheet" would help him. Without hesitation, Ron incorporated the rubber sheet metaphor into his discourse:

So, when you change B and C, it's like taking the graph paper and stretching it and putting it back to normal, or you change A and D, it's like taking the one in the front and stretching it and putting it back to normal, or doing whatever. Stretching it and moving it, do you know what I mean?

In these excerpts, we see Ron resolve relatively easily the question of the changes in B. He proposes that just as the x-axis is moved to explain changes in C, the x-axis is stretched to account for changes in B. The interviewer suggests thinking of this as a separate rubber sheet, so that the changes in the x-axis can be visualized as changes in a plane separate from the plane where the curve is. The implication of this move are very significant, for now he has differentiated the actions of the variables B and C cleanly from those of A and D, as evidenced in the following statement:

B and C... Anything inside the function, stretches and pulls and shrinks it this way or moves it this way [horizontally]. Anything outside the function stretches or pulls it this way [vertically], so now only one rubber sheet changes ....

Moreover, he had an explanation of why different coefficients would be connected to the x- or y-axis:

Because B and C [in $y = Af(Bx + C) + D$] are inside the function. That means you haven't done the function yet, so you're still working on the x-axis. When you're outside the function, you've already done the function and you're working on the y-axis now – directly with the y-axis.

The above excerpt strongly suggests that Ron had thought of horizontal transformations as taking place on the x-axis because "B and C are inside the function" and the vertical ones are made on the graph because "A and D are outside the function." Not only does he use the metaphor of rubber sheets effectively, but he coordinates this metaphor with the metaphor of inside and outside as applied to the symbolic form. This is an example of how algebraic symbolism is not just a matter of manipulating strings of variables from left to right but has a spatial component to it (Kirshner, 1989).

These ideas became clear with the following interaction between Ron and the interviewer, when the former brings into the double rubber sheet metaphor the notions of "thread" and "sticky rubber sheet":

R: Okay, you have two rubber sheets, I have a rubber sheet and a thread, okay?
I: Okay.
R: The thread is the graph which is in the shape of a parabola. B and C, you change B and C so you stretch your rubber sheet ... it's a sticky rubber sheet, okay?
I: Okay, yeah.
R: So you stretch your rubber sheet this way or this way. You push the sticky ... push the string onto the rubber sheet, then you let it go back to normal.
I: Okay, good.
R: Now, whatever you do to this, the string automatically goes with it, okay? So now when I change D, the string automatically goes with it. Because the string is already pressed on there now.
I: Okay, then when I think ... but then, how is that going to move [vertically]? It's already there.
R: So if I shift it up 5, it just goes up 5.
I: So it's not stiff there, it can slide? [referring to whether the graph translates with the x-axis at the same time or whether the axis stays fixed]
R: Yes. No, the whole thing is moving up 5.
I: Well, if the whole thing moves up 5, the graph's not going to go up 5.
R: It just did go up 5.
I: No, it's just going to be like if I cut here and put it up here.
R: Wait. Okay, I see what you're saying. Maybe a row or two rubber sheets would work better.
I: So that's it ... the first step, I like it. You got there, then how do you go from there. When I add 5...
R: I was thinking the... what's... after you did the first part, then you already have the equation, you already have the function on your paper, and then whatever you do the function still goes with it, just like it does here, so when you add 5 to D, it just goes up 5.
I: Hmm hmm.
R: But you're saying that it might work better if you think of it as having the rubber sheet move too, I mean, as having the function move after.
I: Yeah, well, when I see this here, if I think of 2 rubber sheets, I'm getting the clear one that was stuck in the bottom pulled in at 5...
R: Right...
I: My next...

R: Okay, that's better. That'll work.
I: Okay, you know, or if I want a... but I like because you have the tendency to think the opposite way that I do, so that's why I'm happy with it. Because you're thinking of doing everything in the axis, and I'm not doing anything in the axis...

This interaction suggests that Ron used the rubber sheet and a thread as a metaphor. This metaphor worked until vertical movements had to be done. With some help from the interviewer, the double rubber sheet metaphor was brought back in, and Ron made a distinction between "actions in the x-axis" (on the rubber sheet in the back) for the horizontal transformations, and "actions on the graph" (on the front transparent rubber sheet, where the thread was attached) for the vertical transformations.

Ron and the interviewer kept working on the details of the metaphor. "Pins" were brought into the metaphor as a way to link rubber sheets after one of them was stretched. For instance, if the back rubber sheet were stretched by 2 (change in B from 1 to 2), then a transparent rubber sheet with a prototypic parabola on it would be pinned down to it, and both rubber sheets would be released with the elasticity from the back parabola "commanding" the action[10]. Ron then used part of this argument to come up with his "overall generalization":

R: Here's what you do, okay? Here's the overall generalization. You have two rubber sheets, one of them is your graph. The clear one is your... the other one is your function.
I: Yeah, the one on the bottom is the axes.
R: Is the axes.
I: Okay...
R: So anything that is inside the parameters of the functions means you change the graph before you do the function. That means that you're ... that means that B and C are a stretch and a contraction [we assume he confused contraction for translation] before you do your function, okay?
I: So you're going to do that on the graph or the axis?
R: On the graph. Or not on ... on the axes, not on the function, not on the function.
...
R: ... Then when everything is all stretched, you pin it down and you slap our function on top of it, then let everything bounce back the way it should be. That means that when you shift it to the right 5, plop it on, when you bounce it back, it will move it left 5, okay? If you look at your function, shift this over, plop it on, and move the whole thing...
...
R: Plop that on, move it over, okay? Now when you change A and D, what happens is you take your clear one, keep this where it is and you stretch your clear one up this way. Then, when your clear one's all stretched, you plop it on.

In fully developing his approach, Ron moved from actions on the x-axis with "bouncing back", to actions on both axes, to the double rubber sheet metaphor, to the extended double rubber sheet metaphor and its accessories, to the final metaphor. The final metaphor linked the kind of changes that take place in the coefficients to the corresponding changes at the graphical level, consisting of actions on the graph and on the axes. The double rubber sheet solution, which separated actions on the x-axis from those on the y, seems to have solved the issue of coordinating his mouse actions with the plus or minus sign and the $f(x)$ display. It should be noted, however, that in order to do this, Ron constructed a metaphor which contrasted with the intentions of the designers of the software.

DISCUSSION AND CONCLUSION

The three solutions developed by Ron in this case study model the path he pursued among a variety of representations. The excerpts shown in this paper illustrate how the interviewer followed his reasoning, taught and learned with him, and pursued potentially interesting lines of questioning with him. We see such investigations as central to a research program, as we work to delineate the multiple methods one can use to make sense of the topic of transformations of functions.

The data presented illustrate the point that visual reasoning, seeing graphical transformations as movements *on or "of"* the plane, is a potentially powerful form of cognition, and one which requires that we provide students adequate time, opportunities and resources to make constructions, investigations, conjectures and modifications. In times in which high speed computing makes the management of large amounts of data possible, we see such facility in visualization as increasingly important in mathematics education (Eisenberg and Dreyfus, 1991).

Furthermore, we would argue that the strength of Ron's investigation lies in his ability to coordinate those visual actions with changes in other representations. Although the discrepancies among representations generated Ron's problem, he eventually was able to put together his findings across different representations. This investigation supports the view that students can develop effective strategies of inquiry when presented with an environment supporting the use of multiple representations. However, we also wish to emphasize that successful work in such an environment probably depends largely on the careful construction of tasks, the development of didactic sequences and approaches such as the one described herein, and the ability to listen to students. In our work (Confrey, 1992), we have referred to the coordination and contrast among representations as an "epistemology of multiple representations" (Confrey, 1992) and suggested a model of what it means to understand in multiple representational environments (Borba, 1993, 1994a).

The development of the construct of "covariational equations" is postulated to offer an interesting possibility for rethinking what forms of algebraic symbolism to use with students. It can be viewed as either representing a potential alternative form or it can be seen as effective in the transition from the use of patterns of discrete point sets to traditional $f(x)$ notation. This is an example of the value of learning to listen closely to students and then using their "voice" to provide us an opportunity to revise our own perspective in the voice-perspective dialectic (Confrey, 1994b).

As a result of Ron's covariational equations, we proposed a device for coordinating changes in graphical transformations with tabular and algebraic changes (Confrey, 1994a). We displayed the transformations as a series of modifications to the basic prototype. If a prototypic equation was $y = \sin(x)$, then the x's and y's in that relationship can be modified separately to produce a sequence such as the following:

$$\begin{array}{ccc}
y & = & \sin(x) \\
| & & | \\
\text{(mult by 2)} & & \text{(subtract 3)} \\
y' = 2y & & x' = x - 3 \\
| & & | \\
\text{(subtract 3)} & & \text{(mult by 3)} \\
y'' = y' - 3 & & x'' = 3x'
\end{array}$$

To describe y'' in terms of x'', one would have to reverse the changes from x'' to x' to x, then apply the function and move down from y to y' to y''. This gives $y'' = [2\sin((x''/3) + 3)] - 3$. We have found this two-dimensional display an effective curricular intervention with teachers. This illustrates how research on student conceptions (Borba, 1993) can lead to curricular innovation; specifically in this case, for a workshop in teacher education (Smith and Confrey, 1992).

After Ron had achieved a stable system in which he could describe the visual impact of a transformation and its subsequent impact on the covariational equations, the interviewer provided Ron with the challenge of comparing his algebraic form with that of the computer. This episode clearly illustrates how using computers as tools requires semiotic mediation (Wertsch, 1985). The computer software had a specific form of equation built into its design. This form conflicted with the algebraic approaches Ron had developed. As Ron sought to resolve these two forms, he created a new interpretation – one which: 1) was consistent with his previous covariational algebra, 2) allowed him to make sense of the form of algebraic equation displayed on Function Probe, thus resolving a previous discrepancy for him, and 3) brought Ron and the interviewer into a dialogical relationship to yield a visual metaphor of two rubber sheets. In some respects, as a result of an "intershaping relationship" between Ron and the software designers (Borba, 1993, 1995a), not only did the software mediate Ron's actions, but Ron enhanced the possibilities of the software and of the double rubber sheet metaphor as he stretched and translated "axes" and brought thread, tacks and glue into his metaphor.

This case study gives meaning to the claim that new forms of representation change the mathematics to be taught (Confrey, 1993a, 1993b). Mathematics does not exist independently of its representational forms; it exists through those forms. Teaching mathematics with computers is thus not just a matter of placing mathematics onto a machine. Its structure, its processes and its claims for knowledge vary with its medium (Borba, 1994b, 1995b). The investigation we presented here reminds us of the importance of revising our own understanding of mathematics in light of students' initiatives. Design must reflect student methods; curricula must evolve along with design; and student approaches will continue to change as a result of the resources made available to them.

If classrooms are to avoid being left behind in such an evolution, they too must become a critical component of such a loop. Viewing the interview as a specialized form of social interaction, one in which the interviewer generally seeks to understand and follow the method of the student most of the time, and to explore together with him at other times, suggests that the results of the interview studies can inform classroom interactions.

NOTES

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[2] A more thorough discussion of the design can be found in Borba (1993).

[3] To learn about some of Doug's strategies, another student who took part in the same study, see Borba and Confrey (1992), Borba (1993) and Borba (1994a).

[4] This group is directed by Dr. Jere Confrey who was also the designer of FP with Erick Smith (researcher) and Forrest Carrol (leading programmer). The abbreviation FP for Function Probe will be used hereafter.

[5] If the equation of a transformed function is hidden, $y = \ldots$ is shown.

[6] The idea of adding the sampling tool to FP was a result of the development of the design of the study in which Ron was one of the students interviewed, illustrating the interplay between research and software development. A complete discussion of Function Probe can be found in (Confrey et al., 1991).

[7] Ron was both thinking aloud and writing. In the quoted passage, his utterances are represented using algebraic expressions that he actually wrote on scratch paper on day 4.

[8] The reader should take note that the reasoning involved in the last paragraph is fundamental for Ron's third and last solution which will be discussed in the next section.
[9] As Ron develops his metaphor, an ambiguity in language will occur. In some instances, Ron and the interviewer will refer, for example, to the graph of a parabola as being both the parabola and the axes, and sometimes they will refer to just the "curve", in contrast to the axes, as being the graph. Formally, a graph is the former and there is no easy way of labeling the parabola "without a plane and the axes". In this paper, we will continue to label both entities as graph, because we believe that the context helps to clarify what Ron or the interviewer mean at a given time.
[10] There are some operational problems to actually constructing such a metaphor. But as a metaphor, it proved to be very important for Ron and for the interviewer.

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