Statistical properties for a dissipative model of relativistic particles in a wave packet: A parameter space investigation

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\textbf{A B S T R A C T}

Some statistical and dynamical properties for the problem of relativistic charged particles in a wave packet are studied. We show that the introduction of dissipation change the structure of the phase space and attractors appear. Additionally, by changing at least one of the control parameters, the unstable manifold touches the stable manifold of the same saddle fixed point and a boundary crisis occurs. We show that the chaotic attractor is destroyed given place to a transient which follows a power law with exponent $-1$ when varying the control parameters near the criticalities. On the other hand, by changing at least two control parameters and by using the Lyapunov exponents to classify orbits with chaotic and periodic behaviour, we show the existence of infinite shrimp-shaped domains, which correspond to the periodic attractors, embedded in a region with chaotic behaviour. Finally, we show the first indication of a shrimp in a three dimension parameter space.

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\section{1. Introduction}

The theory of Hamiltonian area preserving maps developed very fast during the previous century and many important finding including new theory and results were constructed. Among them lay the well known standard map proposed by Chirikov in 1969 [1,2]. The system describes the dynamics of the kicked rotator and during the years it has been shown that the such a system can be applied in many different fields of science [3–11].

It is well known that the structure of the phase space depends on the individual characteristics of each system and they can be settled in three different classes, namely (i) integrable, (ii) ergodic and (iii) mixed. In case (i) the phase space is filled with straight lines and one example of such a system is the elliptical billiard whose integrability comes from the conservation of the angular momentum with respect to the two foci [12,13]. On the other hand, case (ii) are those systems whose the evolution of a single initial condition is enough the fill the entire phase space. The Bunimovich stadium billiard [14] is one example of such a system. Finally, case (iii), is the most common among them, and in such a system the phase space is composed by Kolmogorov–Arnold–Moser (KAM) islands surrounded by a chaotic sea which is limited by a set of invariant spanning curves [15–19]. The existence of such a tori prevent the unlimited energy growth of the particle [20].

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In this work we consider a system which belongs to case (iii). We revisit the problem of particles moving in a wave packet [21–23] and our main goal is to understand and describe some dynamical properties of the system under the presence of dissipation aiming particularly to describe the dynamics by considering the parameter space. We assume that the particles are moving fast enough, in the sense that Newton’s equation are no longer valid and relativistic corrections must be taken into account. Next, we introduce dissipation into the system by considering that the particles lose a fraction of their energy upon the kicks [24]. Additionally, it is important to mention that the introduction of dissipation changes completely the structure of the phase space in the sense that the chaotic sea might be replaced by a chaotic attractor and elliptical fixed points might be replaced by attracting fixed points each of them with their own basin of attraction. We show that by changing one of the control parameters the edge of the basin of attraction of the attracting fixed points and of the chaotic attractor, which corresponds to the stable and unstable manifolds of a saddle fixed point, touch each other and as a consequence the chaotic attractor as well as its basin of attraction is immediately destroyed. This sudden destruction is called boundary crisis [25,26]. After the boundary crisis, the chaotic attractor is replaced by a transient and we have shown that it can be described by a power law. By changing one of the control parameters, the behaviour of an orbit can change from chaotic to periodic through a boundary crisis. However, what happens if we change at least two of the control parameters simultaneously? To answer this question we revisit the parameter space of the dissipative model of relativistic particles in a wave packet where we show the existence of self-similar structures called shrimps [27–31], which corresponds to the periodic attractors embedded in a chaotic region. Additionally, in order to classify regions in the parameter space with regular or chaotic behaviour we use as a tool the Lyapunov exponent. We adopted the following procedure: starting with a fixed initial condition, after a long transient, the Lyapunov exponent is obtained and for each combination of the control parameters a colour is attributed. After that we give an increment in the parameters and we use the last value obtained for the dynamical variables before the increment, as the new initial condition after the increment. This ensures that we are always in the basin of the same attractor however we lose the information about multiple attractors. Finally, we shown the first evidence of the existence of such a periodic structures in a three dimensional parameter space.

The paper is organised as follows. In Section 2 we introduce the two-dimensional map that describes the dynamics of the system and also we present and discuss our numerical results. Conclusions are drawn in Section 3.

2. The model and the two-dimensional map

In this section we describe the model and discuss our numerical results. We revisit the problem of a relativistic particle in a wave packet, which was introduced by Zaslavsky et al. [21]. Here we have an electrostatic wave packet with constant amplitude $E_0$, a given wave number $\kappa_0$, and an infinite spectrum of harmonic frequencies with separation $\omega$. The relativistic motion of an electron, with a rest mass $m_0$ and charge $-e$ can be described by the following Hamiltonian

$$H(x,p,t) = \sqrt{p^2c^2 + m_0c^4} - \frac{E_0T}{\kappa_0} \cos(\kappa_0 x) \sum_{n=-\infty}^{\infty} \delta(t - nT),$$

where $c$ is the speed of light, the relativistic momentum is given by $p = m_0c/\sqrt{1 - (v/c)^2}$, $T = 2\pi/\omega$ and $n$ gives us the number of kicks in the system. Between the kicks we have free motion, therefore the system can be reduced to a system of two ordinary deferential equation. Following the steps made in [24] it is easy to show that the system can be reduced to a two dimensional map such as

$$P: \begin{cases} I_{n+1} = (1 - \delta)I_n + K \sin(\theta_n), \\ \theta_{n+1} = \theta_n + \frac{I_n}{\sqrt{1 + (\beta I_n)^2}}, \end{cases}$$

where $\theta = \kappa_0 x$, $\beta = \omega / \kappa c$, $I = \kappa_0 p / m_0 c$ and $K = 2\pi E_0 \kappa_0 / m_0 c^2$. Here, $K$ is the parameter controlling the transition from integrability ($K = 0$) to non integrability ($K \neq 0$) and $\beta$ is the parameter that controls the transition from the classic $\beta = 0$ to the relativistic ($\beta \neq 0$) dynamics. Additionally we have introduced a dissipation parameter $\delta \in (0, 1]$ in the sense that the particle loses a fraction of its energy upon the kicks. The introduction of dissipation causes a drastic change the structure of the phase space in the sense that KAM islands are replaced by attracting fixed points and chaotic seas might be replaced by a chaotic attractor.

In order to describe occurrence of homoclinic orbits, we must firstly know exactly the location of the saddle fixed points. Thus, in order to obtain the position of the fixed points we have to solve the equations $I_{n+1} = I_n$ and $\theta_{n+1} = \theta_n + 2m\pi$. The solutions of these equations give us that the saddle fixed points are written as

$$I = \frac{2m\pi}{\sqrt{1 - (\beta 2m\pi)^2}}, \quad \theta = \arcsin \left( \frac{\beta I}{K} \right),$$

where $m = 0, \pm 1, \pm 2, \ldots$. It is well known that a saddle fixed point has both stable and unstable manifolds. The unstable manifolds are obtained by trajectories heading directly to the attracting fixed point (upper branch) or approaching the chaotic attractor (lower branch) in Fig. 1. They are obtained just via the iteration of the mapping $P$ with the appropriate initial conditions. On the other hand, the stable manifolds consist basically of the boundary between the basin of attraction of the attracting fixed point and the basin of attraction of the chaotic attractor. The construction of the stable manifolds are slightly
more complicated than the unstable manifolds, since we have to obtain the inverse of the mapping $P$. The basic procedure is that $P^{-1}(I_{n+1}, \theta_{n+1}) = (I_n, \theta_n)$, thus the expression of the velocity is given by

$$
P^{-1} : \begin{cases}
I_n = \frac{I_n - K \sin(\theta_n)}{(1-\delta)}, \\
\theta_n = \theta_{n+1} - \frac{I_{n+1}}{\sqrt{1+|\mu_n|^2}}.
\end{cases}
$$

Fig. 1(a) shows the behaviour of the stable and unstable manifolds for the saddle fixed point given by Eq. 3 and considering the control parameters $K = 9$, $\delta = 0.4$ and $\beta = 0.15$. However, if one changes one of the parameters in such a way that the unstable manifold touches or even crosses the stable manifold the chaotic attractor is immediately destroyed given place to a transient. Such an event is known as boundary crisis. Here we are interested in characterising the transient or the number of kicks needed until the particle finds its way towards one of the two attracting fixed points. This characteristic time, for a single initial condition can be described by a power law of the type

$$
n_t = \mu_j^n, \quad j = 1, 2, 3,
$$

where $\mu_j$ is written as: (1) $\mu_1 = |\beta - \beta_c|$; (2) $\mu_2 = |\delta - \delta_c|$ and; (3) $\mu_3 = |K - K_c|$ and the index $c$ is used for the critical parameter immediately before the crisis. For the combination of control parameters $\delta = 0.4$ and $K = 9$ we obtained $\beta_c = 0.149128654$, and considering $\beta = 0.15$ and $K = 9$ we found $\delta_c = 0.375598132$, and finally for $\beta = 0.15$ and $\delta = 0.4$ we obtained $K_c = 9.256228541$. It is meaningless to consider the evolution of a single orbit, therefore we need to consider the evolution of an ensemble of initial conditions, thus the average transient is defined as

$$
n_i(t) = \frac{1}{M} \sum_{i=1}^{M} n_i(t),
$$

where the index $i$ denotes a member of an ensemble of initial conditions, and $M$ is the number of different initial conditions. In our simulations we have considered $M = 2 \times 10^4$ initial conditions randomly chosen in the chaotic attractor. Fig. 2 shows the behaviour of the average transient as a function of: (a) $|\beta - \beta_c|$; (b) $|\delta - \delta_c|$ and; (c) $|K - K_c|$ and after a power law fitting we obtained $\rho_1 = -1.05(1)$, $\rho_2 = -1.00(1)$ and $\rho_3 = -1.01(2)$ respectively.

As we have seen, by changing at least one of the control parameters the dynamics can change from chaotic to periodic via boundary crisis. However, what happens if we change simultaneously two or three parameters? As an attempt to answer this question, we revisit the parameter space for the relativistic model of particles in a wave packet. Our main goal is to describe the behaviour of the periodic structures known as shrimp in a two and three dimensional parameter-space. To do so, and as it is usual, we used the Lyapunov exponents [32] as a tool in order to define whether a given region of the parameter space has chaotic or regular behaviour. The Lyapunov exponents are defined as

$$
\lambda_j = \lim_{n \to \infty} \frac{1}{n} \ln |\Lambda_j|, \quad j = 1, 2,
$$

where $\Lambda_j$ are the eigenvalues of $K = \prod_{i=1}^{n} J_i(I_i, \theta_i)$ and $J_i$ is the Jacobian matrix evaluated over the orbit $(I_i, \theta_i)$. If at least one of the $\lambda_j$ is positive then the system is said to have chaotic components. Let us now discuss the behaviour of the parameter space, therefore we might change at least two of the control parameters, namely, dissipation parameter $\delta$, the kicked parameter $K$ and the relativistic parameter $\beta$ of the system (2). For each combination of them we compute the Lyapunov exponent after a long transient and based on this value a colour is attributed. Fig. 3 shows the structure of the parameter-space for the relativistic model of particles in a wave packet. As one can see, the parameter space is very rich of families of periodic structures known as shrimps [27] embedded in a chaotic region. Each shrimp consists of a main body followed by an infinite sequence of bifurcations following the rule, $k \times 2^n$ where the numbers in Fig. 3(a)–(c) correspond to the period of the main structure, namely $k$. The control parameters used in such a figure were: (a) fixed $\beta = 0.2$ and $K \in [11, 11.015]$ and
Fig. 2. Behaviour of the average transient as a function of (a) $|\beta - \delta_0|$ and fixed $\delta = 0.4, K = 9$, (b) $|\delta - \delta_0|$ and $\beta = 0.15, K = 9$ and finally, (c) $|K - K_0|$ for the control parameters $\beta = 0.15$ and $\delta = 0.4$. After a power law fitting we obtained $\rho_1 = -1.05(1), \rho_2 = -1.00(1)$ and $\rho_3 = -1.01(2)$, respectively.

Fig. 3. Parameter space for the model of particles in a relativistic wave packet. Figure (a) was obtained by changing simultaneously $K$ and $\delta$ while (b) was obtained by changing $K$ and $\beta$ and (c) $\beta$ and $\delta$. Figure (d–f) shows the plot of $K$ vs. $\beta$ for the different values of $\delta$ observe that as the value of $\delta$ increases collapse becoming a single body. Figure (g–i), shows the the same region shown in (d–f), however colours represent the period. Figure (j–l) show a shrimp in three different angles, this would be an indications of a periodic structure in a three dimensional parameter space. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article).
the sense that, starting with $I_0 = \theta_0 = 0.1$ as initial condition, for each increment of the combination of the control parameter, we used the last value obtained for $(I, \theta)$ before the increment, as the new initial condition after the increment. In our simulations we have considered a transient of $n = 10^6$ iterations and the Lyapunov exponent were computed for the next $n = 10^6$ iterations. Additionally, the exponents were coded with a continuous colour scale ranging from green–blue (positive exponents) to red–yellow (negative exponents). Fig. 3(d)–(f) shows the $K$ vs. $\beta$ parameter space for the different values of $\delta$, namely $\delta = 0.5196$, $\delta = 0.529173$ and $\delta = 0.541117$ respectively. Observe that, as the value of $\delta$ increases, two structures collapse becoming a single body. Fig. 3(g)–(i), shows the same region shown in (d)–(f), however colours represent the period, namely black corresponds to $k = 3$, red is period 6 and blue denotes period 12. Finally, Fig. 3(j)–(l) show a shrimp of period $k = 4$ in three different angles, this would be an indication of a periodic structure in a three dimensional parameter space.

If we look at one of the periodic structures, for example the one in Fig. 3(a) with $k = 24$, one can see that inside the shrimp there is a region where the Lyapunov exponent is more negative. Additionally, at some point those curves intersect each other. If we define the intersection of such curves as been the head of each shrimp, we can follow its trajectory along the parameter space. Fig. 4(a) shows the position of the head of the shrimp with main period $k = 24$ shown in Fig. 3(a) in the plane $\beta$ vs. $\delta$. After a linear fit one can obtain $\delta = 1.1995 - 2.29\beta$. On the other hand, the plot of $K$ vs. $\beta$ in Fig. 4(b) shows that the position of the head of the shrimp can be described by the function $K = 57.743 - 587.37\beta + 1768.3\beta^2$. So far this is, for the best knowledge of the authors, the first evidence of a shrimp in a 3-D parameter space.

3. Conclusions

We have considered the problem of a charged particle in a wave packet. We have assumed that the particles are moving fast enough and therefore relativistic corrections has been taken into account. Additionally, we have introduced a kicking dissipation into the system where we have assumed that the particles lose some amount of energy upon the kicks. The introduction of dissipation changes the structure of the phase space and attractors appear. As we have shown, by changing at least one of the control parameters, namely, the kicking parameter $K$, the dissipation parameter $\delta$ and the relativistic parameter $\beta$, the unstable manifold touches the stable manifold of a saddle fixed point and as a consequence the chaotic attractor has been destroyed given place to a transient which follows a power law with exponent $-1$ when varying the control parameters near the criticalities. On the other hand, by changing at least two control parameters and by using the Lyapunov exponents as a tool we have shown that the parameter space has a rich structure with infinite shrimp-shaped domains embedded in a region with chaotic behaviour. Finally, we have shown the first indication of a shrimp in a three dimension parameter space.

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