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Characterization of multiple reflections and phase space properties for a periodically corrugated waveguide

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Abstract
Dynamical properties for a beam light inside a sinusoidally corrugated waveguide are discussed in this paper. The beam is confined inside two mirrors: one is flat and the other one is sinusoidally corrugated. The evolution of the system is described by the use of a two-dimensional and nonlinear mapping. The phase space of the system is of mixed type therefore exhibiting a large chaotic sea, periodic islands and invariant KAM curves. A careful discussion of the numerical method to solve the transcendental equations of the mapping is given. We characterize the probability of observing successive reflections of the light by the corrugated mirror and show that it is scaling invariant with respect to the amplitude of the corrugation. Average properties of the chaotic sea are also described by the use of scaling arguments.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The interest in waveguides in the last years has increased and investigations in the subject consider both classical and quantum description as well as experimentation. Applications of the subject take into account many different systems including a technique to detect a large fraction of photons emitted by colour centres within a planar diamond [1], a deterministic scattering of a quantum particle in a two-dimensional ballistic waveguide [2], creation of slow waves near cutoff frequencies [3] and squeezed-light generation in nonlinear planar waveguide [4]. Pulse propagation and electron acceleration in a corrugated plasma is also of interest for waveguides. Historically, direct acceleration of charged particles by electromagnetic fields has
been limited by phase matching, diffraction and material damage thresholds but now scientists have an alternative for producing fast electrons by using plasma micro-optic [5, 6]. Moreover, applications can also extend to transport through a finite GaAs/Al<sub>x</sub>Ga<sub>1-x</sub>As hetero-structure [7], quantized ballistic conductance in a periodically modulated quantum channel [8], quantum transport in ballistic cavities [9], classical versus quantum behaviour in periodic mesoscopic systems [10], underwater acoustics [11–13], anomalous wave transmittance in the stop band of a corrugated parallel-plane waveguide [14] and many other [15].

In this paper we describe the behaviour of a beam of light reflecting specularly from two mirrors. One of them is a flat plane and the other one is periodically corrugate. The dynamics is described by using mappings and considering classical reflections. The formalism used is imported from the billiard<sup>3</sup> dynamics. Therefore, depending on the geometry of the billiard, the phase space falls in one of three classes: (i) regular, (ii) ergodic and (iii) mixed. For the regular case, only periodic and quasi-periodic orbits are observed in the phase space. For the completely ergodic billiards the opposite happens: only chaotic and unstable orbits are present in the dynamics. Mixed structure characterizes the case where either periodic islands, chaotic seas and invariant KAM curves are present in the phase space. The results published in the literature regarding the present model consider so far the so-called simplified version [13, 16–19] (and citations therein). Therefore, in this paper our main goal is to characterize the description of the model taking into account the whole version. Contrary to the simplified version<sup>4</sup> the beam is now let to suffer successive reflections with the corrugated mirror. In our approach, these successive reflections are characterized, to the best knowledge of the authors, for the first time in the literature. They are rare for the full dynamics and the probability of observing a single successive reflection is larger than observing two, that is larger than observing three and so on. We characterize the probability of observing multiple reflections and shown that it is scaling invariant with respect to the corrugation of the mirror. To run the simulations, transcendental equations must be solved. Therefore, we carefully consider four different types of numerical methods to solve the transcendental equations and discuss a qualitative procedure to chose the size of the ensemble to characterize average properties of the phase space. Moreover, with a highly optimized algorithm, we describe average properties of the phase space considering the formalism of scaling function.

The rest of this paper is organized as follows. In section 2, we describe the model and construct the equations of the mapping. The numerical results are considered in section 3 and final remarks and conclusions are drawn in section 4.

2. The model, the mapping and the numerical approach

In this section, we describe all the steps used to construct the equations of the mapping that evolves the dynamics of the problem. A detailed discussion of the numerical method used to obtain the solutions of the transcendental equations, which give the position of the reflection of the beam by the corrugated mirror is also given. The model considered consists of a beam light which is confined to suffer reflections by two mirrors. One of them is flat and located at the position \( y = y_0 \), while the other one is given by the equation \( y = d \cos(kx) \), where \( d \) is the amplitude of the corrugation, \( k \) is the wave number and \( x \) is the position along the mirror. The angles involved in the reflection are obtained assuming a specular reflection. To proceed

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<sup>3</sup> A billiard problem consists of a system in which a point-like particle moves freely inside a bounded region and suffers specular reflections with the boundaries.

<sup>4</sup> In the simplified version, the two mirrors are assumed to be flat but at the instant of the reflection with one of them, the new reflection angle is calculated as if the mirror was corrugated. The approach leads to a good description of the phase space in the domain of small corrugations.
with the construction of the mapping, we note that the system has three control parameters and that the dynamics does not depend on all of them. Therefore, it turns out to define a set of dimensionless and more convenient variables. We define \( X = kx \mod 2\pi \), \( \delta = d/y_0 \) and \( Y = y/y_0 \) so that the corrugated mirror is given by \( Y = \delta \cos(X) \), where \( X \) denotes the phase along the channel, leading also to the flat mirror to be at \( Y = 1 \). A sketch of the model is shown in figure 1.

With this set of new variables, the mapping is given by the variables \( A_n \), denoting the angle of the beam measured from vertical axis (positive clockwise) and the phase of reflection in the corrugated mirror \( X_n \) where the index \( n \) corresponds to the \( n \)th reflection of the beam from the mirrors. Figure 1 shows two typical cases of positive angles of reflection: (1) the reflection occurs after the beam is reflected in the corrugated mirror and (2) the reflection occurs after the beam is reflected from the flat mirror.

For case (1), we assume that the beam starts from the corrugated mirror at the phase \( X_n \). It travels the distance \( \Delta Y = \delta \cos(X_{n+1}) - \delta \cos(X_n) \) and is reflected again. Therefore, \( X_{n+1} \) is obtained numerically by the solution of the transcendental equation

\[
\frac{X_{n+1} - X_n}{\delta \cos(X_{n+1}) - \delta \cos(X_n)} = \tan(A_n). \tag{1}
\]

For case (2), the beam leaves the corrugated mirror \( (X_{n-1}) \) and travels towards the flat mirror. After suffering a specular reflection it travels backwards to the corrugated mirror \( (X_n) \). Similarly, as in case (1), the phase of the reflection is obtained by solving numerically the equation

\[
\frac{X_{n+1} - X_n}{1 - \delta \cos(X_{n+1}) + 1 - \delta \cos(X_n)} = \tan(A_n). \tag{2}
\]
The reflection zone is defined as the region $Y \in [-\delta, \delta]$. Sometimes it is also called the bisection method.
Figure 2. Working lines and two example beams are shown, one to the left and another one to the right of the working lines for: (a), (b) where there are beams coming from the flat mirror; (c) where there are beams coming from the corrugated mirror. $X_0$ and $X_l$ are the intersection, respectively, of the beam and the working line with the line $Y = \delta$.

the orbit in sequence has no physical meaning, therefore the accuracy of the numerical method is of extreme importance.

Although the Newton–Raphson method (see [20] for specific discussions) $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$, with $i = 0, 1, 2, \ldots$, does not have global convergence, we show that it can be used in the present problem with the appropriate choices of initial conditions. In the Newton–Raphson method, the monotonic convergence is guaranteed if in the interval brackets the first-order derivative exists and is non-zero and the second-order derivative of the function exists and does not change sign. Moreover, such sign must be the same as the function.

Given a beam is coming from a reflection with the flat mirror (case (2)) and with positive angle $B_n$ which is less than the angle (measured like $A_n$ and $B_n$) of the tangent in the inflection point $(\pi/2)$ (of the corrugated mirror), we can find the lowest phase $(0, \ldots, 2\pi)$ in which a working line tangent to the corrugated mirror is defined that has an angle equal to the $B_n$ of beam (figure 2(a)). If the angle $B_n$ of the beam is greater, we consider the working line of the same beam angle and crossing x-axis at phase $(\pi/2)$ (figure 2(b)). Styling, $X_i$, the intersection phase of working line with lines $Y = \delta$ and $X_0$, the re-entry phase of beam in the reflection zone is defined as earlier. If $X_i \geq X_0$, the Newton–Raphson method converges monotonically from $X_0$ to the phase of the next reflection $X_1$. However, if $X_i < X_0$, the Newton–Raphson method converges monotonically from the phase $3\pi/2$ (inflection of the mirror) to the new reflection phase.

Similarly, when the beam comes from a reflection at the corrugated mirror (case (1)) with positive angle $B_n$ greater than the angle of the tangent in the inflection point $(3\pi/2)$, we can find the largest phase $(0, \ldots, 2\pi)$, where the working line tangent to the corrugated mirror
and with angle equal to the angle of the beam can be defined (see figure 2(c)). If $X_l \leq X_0$, the Newton–Raphson method converges monotonically from phase $3\pi/2$ (inflection of the mirror). If either $X_l > X_0$ or the angle of the beam is smaller than the angle of the tangent line in phase $3\pi/2$ (inflection of the mirror) or if $X_n \geq 3\pi/2$, the beam goes to the flat mirror without being reflected again at corrugated mirror.

For case (1), when the angle $B_n$ of the beam is greater than or equal to $\pi/2$, certainly there will be a new reflection at the corrugated mirror and the Newton–Raphson method converges monotonically from the inflection phase $3\pi/2$. Similar treatment is given to the case of negative angle $B_n$ of beam.

Various methods derived from Newton–Raphson method (polishing method) can improve the factor or even the order of the convergence. They can even circumvent some limitations or difficulties in their use. One of them is the requirement of non-zero derivative in the interval bracketing the root, but do so at the expense of greater complexity in the iterations. We consider the use of the secant method which makes approximation of the derivative, $f'(x)$, using secant line by the two last approaches. This method has order of convergence $P \approx 1.62$ (golden ratio) against $P = 2$ of the Newton–Raphson method [20]. However, each iteration requires only evaluation of the function itself while the Newton–Raphson method requires the calculation of the function and its derivative. It can be far more complex than the function itself. It is not the case in the problem at hand.

To use the secant method we must first obtain one more approach necessary for the start of the method. For this, we used the Newton–Raphson method. For illustration purposes, we present in figure 3, for $\delta = 0.1$, the relative frequency histograms of the number of iterations of the numerical methods required to achieve the beam’s phase reflection in the corrugated mirror with an accuracy of $10^{-13}$ using IEEE double precision floating point representation.
One can see an average number of iterations by reflection of 4.76 for the secant method, 5.42 for the regula-falsi method and 34.83 for the Bolzano method. The interval testing was considered as of size $2\pi/500$. With such size, an extensive search for the bracketing the first solution leads to an average of 42.09 searches. The function is evaluated at each step of the search for interval that brackets the first root, so the average number of function evaluations is 4.76 for the secant method, 47.51 for the regula-falsi method and 76.92 for the Bolzano method.

As the computational performance depends of several other factors, the elapsed times of five runs were measured and the averages and standard deviations of the observations were made and expressed as a ratio of time spent by the Bolzano method. Our results are $T_{\text{secant}}/T_{\text{Bolzano}} = 0.139(4)$ and $T_{\text{regula-falsi}}/T_{\text{Bolzano}} = 0.614(5)$.

3. Numerical results

In this section we present our numerical results for the model. After the description of the method used to solve the transcendental equations, the phase space can be obtained. To do so, it is more convenient to use the angle of beam’s reflection as $\theta = \pi/2 - A$. Figure 4(a) shows the phase space obtained for a light beam in a periodically corrugated waveguide for the control parameter $\delta = 0.1$ and in figure 4(b) for $\delta = 0.01$. One sees that a mixed structure is present in the sense that periodic islands marked by a resonance (elliptic fixed point), chaotic seas and invariant KAM curves can all be observed. The size of the chaotic sea and the position of the invariant KAM curves depend on the value of the control parameter $\delta$ and as discussed in [17], the position of the invariant KAM curve scales with $\delta^{1/2}$.

During the dynamics of the light, a special kind of reflection can be observed, namely a successive reflection. Such type of reflections are indeed defined as being reflections of the light with the corrugated part of the mirror from the light coming from the same corrugate part. These events are rare compared with the whole dynamics. Then the probability of observing one successive reflection is larger than observing two, which is large than observing three and so on. An estimated (for $10^{10}$ iterations) probability distribution ($P$) of these successive reflections is shown in figure 5(a) for different control parameters, as labelled in the figure. The behaviour of $P$ is given by a power law of the type $P \propto n_{sr}^\gamma$ with a numerical fitting giving $\gamma = -3.76(3)$ where the index $sr$ denotes successive reflections. Considering that all the curves have the same slope, this is an indication that the behaviour of $P$ could be scaling
invariant with respect to different control parameters. To check this, we chose a particular probability, say $10^{-6}$ and obtain the corresponding number of successive reflections, at that frequency ($n'$), for different values of control parameter $\delta$. Figure 6 shows a plot of $n'_s \times \delta$ where a power law fitting furnishes a slope $0.252$. By rescaling ($n_s \rightarrow n_s/\delta^{0.252}$), all the curves of $P$ overlap each other onto a single curve, as shown in Figure 5(b). This is a confirmation that $P$ is scaling invariant with respect to the control parameter $\delta$.

Let us now discuss some properties of the phase space, more specifically for the chaotic orbits. The most natural observable to look at in the chaotic orbits is the average angle of reflection. It is defined as

$$\overline{\theta}(n, \delta) = \frac{1}{n} \sum_{i=1}^{n} \theta_i.$$  \hspace{1cm} (3)

From this definition, the deviation of average angle is given by

$$\omega(n, \delta) = \frac{1}{M} \sum_{j=1}^{M} \sqrt{\overline{\theta}_j^2(n, \delta) - \overline{\theta}^2(n, \delta)},$$  \hspace{1cm} (4)

where $M$ denotes an ensemble of different initial conditions, all of them chosen along the chaotic sea. The main goal is to describe the behaviour of $\omega$ as a function of $(n, \delta)$. Before evolving the simulations, let us first discuss the size of the ensemble. To have a good statistics.
Figure 7. Plot of $\Delta \omega / \omega \times M$ for: (a) $n = 10$ collisions; (b) $n = 10000$ collisions. The same behaviour was obtained for several other values of $n \in [10, 10000]$.

Figure 8. (a) Plot of $\omega \times n$ for the parameters $\delta = 2 \times 10^{-4}$, $\delta = 5 \times 10^{-4}$ and $\delta = 3 \times 10^{-3}$. (b) After a rescaling of $(n \rightarrow n \delta^2 + z$ and $\omega \rightarrow \omega / \delta^\alpha$) all curves of (a) are overlapped onto a single plot. We used an ensemble of $M = 1000$. We conclude that $M = 500$, initial conditions generate an ensemble sufficiently large to stabilize the metric $R$ for any $n \in [10, 10000]$. For safety, in our simulations, we consider $M = 1000$. For $\omega$ but at the same time have a reasonable computational time, we investigate initially the behaviour of the ratio of the standard deviation, $\Delta \omega$, over their ensemble average $\omega$, i.e. $R = \Delta \omega / \omega$. Figure 7 shows a plot of $R$ as a function of the size of the ensemble. We conclude that $M = 500$, initial conditions generate an ensemble sufficiently large to stabilize the metric $R$ for any $n \in [10, 10000]$. For safety, in our simulations, we consider $M = 1000$.

A typical plot of $\omega \times n$ is shown in figure 8(a). One sees that $\omega$ starts growing according to a power law and eventually, after reaching a characteristic crossover time $n_\chi$, it bends towards a regime of saturation for long enough $n$. As discussed previously for the simplified case [16], the following scaling hypotheses can be proposed:

(i) For $n \ll n_\chi$, the behaviour of $\omega$ can be described as

$$\omega(n \delta^2, \delta) \propto \left[n \delta^2 \right]^\beta,$$

where $\beta$ is a critical exponent.

(ii) For $n \gg n_\chi$, $\omega_{\text{sat}}$ is given by

$$\omega_{\text{sat}}(n \delta^2, \delta) \propto \delta^\alpha,$$

where $\alpha$ is a critical exponent.
Figure 9. For $M = 1000$: (a) plot of $\omega_{\text{sat}} \times \delta$ where the critical exponent found was $\alpha = 0.508(4)$. (b) Results for $n_x \times \delta$, where $z = -0.956(9)$.

(iii) The characteristic crossover $n_x$ is written as

$$n_x(n^2, \delta) \propto \delta^z,$$

and $z$ is the dynamic exponent.

The critical exponents can be obtained if the behaviour of $\omega_{\text{sat}}$ and $n_x$ are obtained as a function of the control parameters. From our numerical investigations, we found that $\beta \cong 0.5$. The critical exponents obtained as a function of $\delta$ are shown in figures 9(a) and (b). Power law fittings produce $\alpha = 0.508(4)$ and $z = -0.956(9) \cong -1$. Given the critical exponents are known, a rescale to the axis can be made therefore producing a single and universal plot for different curves of $\omega$, as shown in figure 8(b).

Let us now comment on the critical exponents obtained from figures 9(a) and (b). The system is characterized by a control parameter $\delta$. In fairness, if $\delta = 0$, the system is integrable therefore leading to the observation of only periodic and quasi-periodic orbits in the phase space. As far as $\delta \neq 0$, the phase space is born as mixed showing, among the regular parts, a chaotic portion. Average properties for the chaotic orbits are characterized by power laws and they are also described by a homogeneous and generalized function [21]. The set of critical exponents obtained is $\alpha = 0.5, \beta = 0.5$ and $z = -1$. Surprisingly, these exponents are also the same as those observed for the so-called one-dimensional Fermi accelerator model [22]. The accelerator model consists of a classical particle of mass $m$ which is confined to bounce with two walls. One of them moves periodically in time and the other one is fixed and works as a returning mechanism for the particle to suffer a further collision with the moving wall. Collisions of the particle with either walls are assumed to be elastic. Although the set of variables involved in the two distinct problems (Fermi accelerator and periodically corrugated waveguide) are different, the phase transition observed from integrability to non-integrability leads one to observe the same set of critical exponents; therefore, the two different models belong to the same class of universality near such a transition.

4. Summary and conclusions

As a short summary, we have characterized some dynamical properties for a corrugated waveguide considering the description of the model by the use of a complete version for the
mapping. Therefore, successive reflections of the beam with the corrugated mirror are allowed. We have shown that the phase space of the model is of mixed type that includes periodic islands, chaotic seas and invariant KAM curves separating different regions of chaos. The numerical simulations considered the secant method to solve the transcendental equations which produced remarkably fast simulations leading to a spent time of the order of 13.9% of time required by the Bolzano method. The successive reflections of the beam with the corrugated mirror were characterized by obtaining their probability, which is scaling invariant with respect to the control parameter. Finally, average properties of the chaotic sea considering the possibility of having successive collisions lead us to obtain the same set of critical exponents observed for the simplified case [16]. Therefore, the curves of deviation of the average angle are also scaling invariant with respect to the control parameter $\delta$ and the set of critical exponent obtained makes the present model to belong to the same class of universality of the one-dimensional Fermi accelerator model [22].

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