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Critical exponents for a transition from integrability to non-integrability via localization of invariant tori in the Hamiltonian system

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Abstract
Critical exponents that describe a transition from integrability to non-integrability in a two-dimensional, nonlinear and area-preserving map are obtained via localization of the first invariant spanning curve (invariant tori) in the phase space. In a general class of systems, the position of the first invariant tori is estimated by reducing the mapping of the system to the standard mapping where a transition takes place from local to global chaos. The phase space of the mapping shows a large chaotic sea surrounding periodic islands and limited by a set of invariant tori whose position of the first of them depends on the control parameters. The formalism leads us to obtain analytically critical exponents that describe the behaviour of the average variable (action) along the chaotic sea. The result is compared to several models in the literature confirming the approach is of large interest. The formalism used is general and the procedure can be extended to many other different systems.

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(Some figures in this article are in colour only in the electronic version)
The interest in mappings is because of the possible applications in the study of channel flows [11, 12], waveguide [13], transport properties [14], Fermi acceleration [15–19] and also for the study of magnetic field lines in toroidal plasma devices with reversed shear (like tokamaks) and many other [20–22].

Generally, the mappings have one or more control parameters. They control the intensity of the nonlinearity, as well as a transition from integrability to non-integrability. The phase space mostly exhibits mixed forms in the sense that a large chaotic sea exists and eventually surrounds periodic islands [23, 24] and is limited by a set of invariant tori. The size of the chaotic sea is strongly influenced by the control parameters since they play the role of controlling the intensity of the nonlinearity.

In this communication, we concentrate specifically in the localization of the first invariant tori (sometimes also called as the invariant spanning curve) in the phase space aiming to obtain, via an analytical approach, the critical exponents which describe the average behaviour of the observables along the chaotic sea for a two-dimensional mapping. We used a connection with the standard mapping [23], near a transition from local to global chaos, to obtain an expression for an effective control parameter. The formalism developed can be extended to many other different kinds of mappings.

In order to construct the mapping, we assume that there is a two-dimensional integrable system that is perturbed. In generalized coordinates, the controlling Hamiltonian is written as

\begin{equation}
H(J_1, J_2, \theta_1, \theta_2) = H_0(J_1, J_2) + \epsilon H_1(J_1, J_2, \theta_1, \theta_2),
\end{equation}

where the variables \(J_i\) and \(\theta_i\) with \(i = 1, 2\) correspond, respectively, to the action and the angle. Here, \(H_1\) and \(H_0\) describe the interacting and non-interacting Hamiltonian of the system. One sees that the control parameter \(\epsilon\) defines the magnitude of coupling and controls a transition from integrability to non-integrability. To use the characterization of the dynamics in terms of a mapping, we consider a Poincaré section defined by the plane \(J_1 \times \theta_1\) and assume \(\theta_2\) as constant (mod \(2\pi\)). A two-dimensional map which qualitatively describes the behaviour of (1) is

\begin{equation}
T : \begin{align*}
J_{n+1} &= J_n + \epsilon h(\theta_n, J_{n+1}) \\
\theta_{n+1} &= [\theta_n + F(J_{n+1}) + \epsilon p(\theta_n, J_{n+1})] \text{mod}(2\pi),
\end{align*}
\end{equation}

where \(h, F\) and \(p\) are assumed to be nonlinear functions of their variables while the index \(n\) denotes the \(n\)th iteration of the mapping. Since the map, in equation (2), is to be area preserving, the expressions for \(h(\theta_n, J_{n+1})\) and \(p(\theta_n, J_{n+1})\) are related as

\begin{equation}
\frac{\partial p(\theta_n, J_{n+1})}{\partial \theta_n} + \frac{\partial h(\theta_n, J_{n+1})}{\partial J_{n+1}} = 0.
\end{equation}

When the function \(F\) diverges in the limit of the vanishing action, generally the phase space exhibits the mixed form with the coexistence of periodic islands, chaotic seas and invariant tori. Therefore, the formalism we develop is based on the consideration that near the invariant spanning curves the dynamics can be reduced locally to the standard mapping [23]. For simplicity, we take \(h(\theta_n) = \sin(\theta_n)\) and have liberty to chose \(p\) and \(F\). Without losing generality, for a wide class of systems we may consider \(p(\theta_n, J_{n+1}) = \text{constant}\) and vary \(F\). The systems within the scope of the general two-dimensional map include the logistic twist mapping [25], the Taylor–Chirikov map [26], the Fermi–Ulam accelerator model [27], the Fermi–Putylnikov accelerator model [28, 29] and the Hybrid–Fermi–Ulam–bouncer model [30].

We suppose that near an invariant tori, which limits the size of the chaotic sea, \(J\) can be written as \(J_{n+1} \equiv J^* + \Delta J_{n+1}\), where \(J^*\) is a typical value along the invariant tori and \(\Delta J_{n+1}\) is
a small perturbation to $J_{n+1}$. This approximation can be used because the invariant spanning curve fluctuates very little as compared to the whole chaotic sea. Hence, the second equation of mapping (2) is rewritten as

$$\theta_{n+1} = \theta_n + F(J^* + \Delta J_{n+1}).$$

(4)

Expanding the nonlinear function $F(J^* + \Delta J_{n+1})$ in the Taylor series and keeping only terms of first order, we have

$$\theta_{n+1} = \theta_n + F(J^*) + \Delta J_{n+1} F'(J^*).$$

(5)

where $F' = dF/dJ$. Following the same procedure, the first equation of equation (2) is written as

$$I_{n+1} = I_n + \epsilon F'(J^*) \sin(\theta_n).$$

(6)

After these transformations, equation (2) becomes

$$T : \begin{cases} 
\theta_{n+1} = \left[\theta_n + I_{n+1}\right] \mod (2\pi) \\
I_{n+1} = I_n + F'(J^*) \epsilon \sin(\theta_n)
\end{cases},$$

(7)

which is the Chirikov–Taylor map. Here, the effective control parameter $K_{eff}$ is

$$K_{eff} \cong F'(J^*) \epsilon.$$  

(8)

This result confirms that the formalism is generic and can be applied, in principle, to a wide class of functions $F(J)$. Hence, near the invariant spanning curves, we can reduce the system mapping in general to the Chirikov–Taylor map for a wide range of systems. The formalism so developed, as discussed later, helps us to calculate the location of the invariant spanning curve and at the same time enables us to obtain the effective control parameter near the first invariant spanning curve and critical exponents.

To illustrate the applicability of the formalism, let us first consider the case

$$F(J) = \frac{1}{J^\gamma},$$

(9)

where $0 < \gamma < 1$ is a control parameter. For this case, we obtain

$$F'(J^*) = -\frac{\gamma}{J^{\gamma+1}}.$$  

(10)

Since the transition from local to global chaos occurs at $K_{eff} \cong 0.9716 \ldots$ (see [23] for specific details), the average value at the invariant tori is given by

$$J^* \cong \left(\frac{\gamma \epsilon}{0.9716 \ldots}\right)^{1/(\gamma+1)}.$$  

(11)

As an extension of this procedure, we can conclude that the size of the chaotic sea is proportional to $(\gamma \epsilon)^{1/(\gamma+1)}$. The phase space generated from mapping (2) using $F(J)$ as described in equation (9) is shown in figure 1. One sees that the phase space is mixed and contains a set of periodic islands that are surrounded by a large chaotic sea that is limited by a set of invariant tori. The control parameters used were $\epsilon = 10^{-3}$ and (a) $\gamma = 2/5$ and (b) $\gamma = 1/2$. We noted that the size of the chaotic sea varies as the control parameter $\gamma$ varies. Consequently, average properties of the dynamics are also dependent on the position of the invariant spanning curves. The behaviour of $J$, as shown in figure 2(a) is described as (i) $J \propto \epsilon^\alpha$ for $n \gg n_c$; (ii) $J \propto n^\beta$ for $n \ll n_c$; (iii) $n_c \epsilon^z \propto \epsilon^\beta$ where $n_c$ denotes the crossover iteration number that marks the change from the regime of growth to the regime of saturation. $\alpha$, $\beta$, and $z$ are critical exponents. The variable $J$ corresponds to the average of the action along the chaotic sea averaged from two different procedures: (1) first, we consider the average over the orbit defined as $J_i = 1/n \sum_{k=1}^{n} J_k$ and then (2) average $J_i$ over and ensemble
of \( M \) different initial conditions as \( \bar{J} = 1/M \sum_{i=1}^{M} J_i \). The initial conditions were chosen such that \( J_0 \) was kept fixed as \( J_0 = 10^3 \epsilon \) and an ensemble of \( M = 5 \times 10^3 \) randomly chosen angle in the interval \( \theta \in [0, 2\pi) \). The small initial action was chosen to avoid a second and additional crossover (see [31] for a specific discussion of this crossover and applications in the Fermi–Ulam model).

Let us now suppose that \( \bar{J} \) is dependent on the position of the lowest invariant spanning curve, as foreseen by equation (11), which leads us to conclude that

\[
\alpha \approx \frac{1}{1 + \gamma}.
\]

Using scaling arguments (similarly as done in [24]), one can show that \( z = \alpha/\beta - 2 \), therefore leading to

\[
z \approx \left[ \frac{1}{\beta(1 + \gamma)} - 2 \right].
\]

To validate equations (12) and (13), several numerical simulations were made for different sets of control parameters and a large ensemble of initial conditions. Our results give that \( \beta \approx 0.5 \) while the critical exponents are shown in table 1. A plot of the critical exponents obtained by numerical simulations is shown in figure 3 together with the analytical result, where the agreement between the results is fairly well.
Figure 2. (a) Plot of $J$ vs $n\epsilon^2$. (b) Merger of all curves of (a) onto a single and universal plot. $\gamma$ was fixed as $\gamma = 1/2$.

Table 1. Comparison of the critical exponent $1/(1 + \gamma)$ and $\alpha$, $1/[\beta(1 + \gamma)] - 2$ and $z$. The range considered was $\epsilon \in [10^{-5}, 10^{-3}]$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$1/(1 + \gamma)$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$1/[\beta(1 + \gamma)] - 2$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3/7$</td>
<td>$7/10$</td>
<td>0.696(4) $\approx 7/10$</td>
<td>0.487(7)</td>
<td>$-0.56(2)$</td>
<td>$-0.57(2)$</td>
</tr>
<tr>
<td>$4/9$</td>
<td>$9/13$</td>
<td>0.710(2) $\approx 9/13$</td>
<td>0.484(5)</td>
<td>$-0.57(2)$</td>
<td>$-0.58(1)$</td>
</tr>
<tr>
<td>$1/2$</td>
<td>$2/3$</td>
<td>0.673(2) $\approx 2/3$</td>
<td>0.488(4)</td>
<td>$-0.63(2)$</td>
<td>$-0.641(7)$</td>
</tr>
<tr>
<td>$3/5$</td>
<td>$5/8$</td>
<td>0.591(1) $\approx 5/8$</td>
<td>0.488(5)</td>
<td>$-0.71(1)$</td>
<td>$-0.68(2)$</td>
</tr>
<tr>
<td>$2/3$</td>
<td>$3/5$</td>
<td>0.607(1) $\approx 3/5$</td>
<td>0.489(7)</td>
<td>$-0.77(2)$</td>
<td>$-0.757(4)$</td>
</tr>
<tr>
<td>$5/7$</td>
<td>$7/12$</td>
<td>0.5893(9) $\approx 7/12$</td>
<td>0.491(5)</td>
<td>$-0.81(1)$</td>
<td>$-0.808(5)$</td>
</tr>
<tr>
<td>$3/4$</td>
<td>$4/7$</td>
<td>0.588(3) $\approx 4/7$</td>
<td>0.488(6)</td>
<td>$-0.82(1)$</td>
<td>$-0.817(8)$</td>
</tr>
<tr>
<td>$4/5$</td>
<td>$5/9$</td>
<td>0.563(1) $\approx 5/9$</td>
<td>0.489(7)</td>
<td>$-0.86(3)$</td>
<td>$-0.858(6)$</td>
</tr>
<tr>
<td>$1$</td>
<td>$1/2$</td>
<td>0.518(4) $\approx 1/2$</td>
<td>0.495(6)</td>
<td>$-0.98(1)$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>
Figure 3. Plot of the critical exponents (a) $\alpha$ vs $\gamma$ and (b) $z$ vs $\gamma$. Circles are the numerical results and lines correspond to the analytical result.

We see that after a proper rescaling of the axis with the critical exponents, all the curves of $\bar{J}$ overlap each other onto a single and universal plot, as shown in figure 2(b), therefore confirming the scale invariance of the chaotic sea.

Extensions of the formalism can, in principle, be made for different models as well. To illustrate few examples, for the case of $\gamma = 1$ and considering the transformations $J \rightarrow V$ and $\theta \rightarrow \phi$, one can recover the Fermi–Ulam accelerator model [31]. Considering $J \rightarrow \gamma$ and $\theta \rightarrow X$ where $\gamma$ in this transformation represents the angular coordinate instead of the control parameter (as is the case for this communication), one can have the periodically corrugate waveguide [13]. In these two cases, the critical exponents obtained were $\alpha = 0.5$ and $z = -1$ which is very well predicted by equations (12) and (13) and shown in table 1. On the other hand, for the case of $\gamma = 1/2$, a similar map describes the dynamics of a classical particle confined inside an infinitely deep box of potential containing a periodically moving square well [32, 33] or time varying barrier [34]. For these cases, the critical exponent is $\alpha = 2/3$.

As a summary, we studied in this work the localization of the invariant tori that characterizes the transition from local to globally chaotic dynamics for a two-dimensional, area-preserving Hamiltonian map. The procedure used consists in describing locally (near the first invariant spanning curve) the dynamics by using the standard mapping. Critical exponents describing a transition from integrability to non-integrability were obtained analytically and compared with numerical results for at least four different models. The procedure used showed to be of large interest and is extensible to many other systems.
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