Scaling properties of the regular dynamics for a dissipative bouncing ball model

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Abstract

Some scaling properties of the regular dynamics for a dissipative version of the one-dimensional Fermi accelerator model are studied. The dynamics of the model is given in terms of a two-dimensional nonlinear area contracting map. Our results show that the velocities of saddle fixed points (saddle velocities) can be described using scaling arguments for different values of the control parameter.

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1. Introduction

The one-dimensional Fermi accelerator model consists basically in a classical particle confined in and bouncing between two rigid walls. One of them is assumed to be fixed while the other one is periodically time varying. The origin of such a model backs to early 1949 when Fermi [1] attempted to explain the cosmic rays acceleration. After that, Ulam [2] gave a mathematical support for the model proposing dynamical equations that could be approximated considering the formalism of discrete mappings. Moreover, the system has been considered as a toy model and the mathematical as well as the numerical tools developed to study the model have been widely used in many other dynamical systems described by using maps. Recently, with the advent of fast computers, the model has received much attention and was studied [3–9] in different versions and under distinct approaches [10–17]. One of the most important property of the non-dissipative version (dissipative forces are absent) is that it has a mixed phase space structure. Thus, one can observe invariant spanning curves [6,18–20] that limit the size of the chaotic sea which is characterized by a positive Lyapunov exponent.

In this paper, we revisit the one-dimensional Fermi accelerator model under the presence of inelastic collisions seeking to understand and describe a scaling behavior present in the velocity of saddle fixed points. For a two-dimensional mapping, a saddle fixed point is defined as being that whose eigenvalues of the linearized map show that $|\lambda_1| > 1$ and $|\lambda_2| < 1$. In our approach, we have introduced dissipation via inelastic collisions with the two walls. The phase space now shows the property of measure contraction. It is important

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to emphasize that the contraction of the phase space measure, caused by inelastic collisions, has profound consequences on the dynamics in physical experiments [21–23]. Particularly, this kind of damping allows one to extract many important properties of the system including chaotic attractors [24–26], and even crisis events can be observed and characterized [27–29].

In the present paper we are interested in the description of the behavior of saddle velocities as function of the control parameters. Using scaling arguments, it is possible to collapse all the saddle velocity curves onto a single and universal plot, confirming that the saddle velocities are scaling invariant. The paper is then organized as follows. In Section 2 we describe the mapping and discuss briefly some analytical properties. Section 3 is devoted to obtain the scaling results when concluding remarks are drawn in Section 4.

2. A dissipative bouncing ball model and the mapping

A dissipative version of the bouncing ball model that we are considering1 consists of a classical particle moving horizontally. Additionally, it is confined in and bounces inelastically two parallel and rigid walls. One of the walls is assumed to be fixed at \( x = l \) while the other one is periodically time varying, i.e. \( x_n(t) = \epsilon \cos(\omega t) \). Here \( \epsilon \) denotes the amplitude of oscillation and \( \omega \) is the angular frequency. We shall consider that when the particle hits the fixed wall, it experiences an inelastic collision given by a restitution coefficient \( \alpha \). For \( \alpha = 1 \), the collision with the fixed wall is elastic and the particle returns to the moving wall with the same kinetic energy. On the other hand, for \( \alpha = 0 \) the particle suffers a completely inelastic impact, and just a single collision is enough to terminate all the particle’s dynamics. This case is not of interest in the present study. Therefore we are considering values for \( \alpha \in (0, 1) \). Moreover, we assume that the restitution coefficient of the moving wall is \( \beta \). For the case of \( \beta = 0 \), the particle is relaunched from the periodic time-varying wall with the maximum value of the moving wall’s velocity. Thus, for such limit of \( \beta \) it is possible to observe the phenomenon of locking [30,31]. The case where \( \beta = 1 \) yields elastic collisions with the moving wall. Again, we will consider \( \beta \in (0, 1) \).

As it is so usual in the literature, the dynamics of the model is described by a two-dimensional non-linear mapping on the variables \( V \) and \( \phi \), where \( V \) is the particle’s velocity and \( \phi \) is the phase of the moving wall. After defining the following dimensionless variables \( \varepsilon = \epsilon/l, V = V/(\omega l) \) and measuring the time in terms of the number of oscillations of the moving wall \( \phi = \omega t \), we obtain that the mapping which describes the dynamics is given by

\[
T: \begin{cases}
V_{n+1} = V_n - (1 + \beta)\varepsilon\sin(\phi_{n+1}), \\
\phi_{n+1} = (\phi_n + \Delta T_n) \mod (2\pi),
\end{cases}
\]

(1)

where the index \( n \) denotes the \( n \)th collision with the moving wall. The expressions for the velocity \( V_n \) and the time \( \Delta T_n \) depend on what kind of collision the particle is experiencing, namely (i) multiple hits with the moving wall or (ii) a single hit with the time-varying wall. For the case (i), the expressions are \( V_n = -\beta V_n \) and \( \Delta T_n = \phi_c \) with \( \phi_c \) obtained from the smallest numerical solution of \( G(\phi_c) = 0 \) where

\[
G(\phi_c) = \varepsilon \cos(\phi_c) - \varepsilon \cos(\phi_n) - V_n \phi_c.
\]

The function \( G(\phi_c) \) is obtained from the condition that matches the same position for both the particle and the moving wall. If the particle suffers a multiple collision before exiting the collision zone,\(^2\) then we obtain necessarily that \( G(\phi_c) = 0 \) for \( \phi_c \in (0, 2\pi] \). On the other hand, i.e. for the case (ii), if the particle leaves the collision zone after suffering a single collision, then it travels toward the direction of the fixed wall, suffers an inelastic collision with such a fixed wall and, after having a fractional loss of energy, it is reflected backwards to the moving wall. The corresponding expressions are \( V_n = \beta V_n \) and \( \Delta T_n = \phi_c + \phi_l + \phi_1 \). The auxiliary terms are \( \phi_i = (1 - \varepsilon \cos(\phi_n))/V_n, \phi_f = (1 - \varepsilon)/(\alpha V_n) \), and finally, the term \( \phi_c \) is numerically obtained from the smallest solution of \( F(\phi_c) = 0 \) with

\[
F(\phi_c) = \varepsilon \cos(\phi_n + \phi_f + \phi_i + \phi_1) - \varepsilon + \alpha V_n \phi_c.
\]

\(^1\)Sometimes the bouncing ball model is also called as Fermi–Ulam model (FUM).

\(^2\)The collision zone is defined as the region \( x \in [-\epsilon, \epsilon] \).
The equation $F(\phi_c) = 0$ gives the instant of the impact of the particle with the moving wall. It was obtained as an attempt to account the condition that, at the instant of the hit, the particle is at the same position of the moving wall.

A careful investigation of the mapping (1) together with some algebra allows us to show that for the case (i), the determinant of the Jacobian matrix is

$$\det J = \beta^2 \left[ \frac{V_n + e \sin(\phi_n)}{V_{n+1} + e \sin(\phi_{n+1})} \right],$$

while for a similar procedure for the case (ii) yields

$$\det J = \alpha^2 \beta^2 \left[ \frac{V_n + e \sin(\phi_n)}{V_{n+1} + e \sin(\phi_{n+1})} \right].$$

The above results thus confirm that the measure preservation of the phase space is recovered for both cases only for $\beta = \alpha = 1$.

Let us now discuss the procedures used to obtain the fixed points. They are obtained from the condition that $V_{n+i} = V_n$ and $\phi_{n+i} = \phi_n$ simultaneously. If one makes $i = 1$, there is a period one fixed point; if $i = 2$, there is a period two fixed point and so on. The two equations that must be solved are

$$V_{n+1} = \alpha \beta V_n - (1 + \beta) e \sin(\phi_{n+1}) = V_n,$$

$$\phi_{n+1} = \phi_n + \phi_r + \phi_1 + \phi_c = \phi_n + 2m\pi$$

with $m = 1, 2, 3 \ldots$ denoting the number of complete oscillations of the moving wall.

Solution of Eqs. (2) and (3) yields that

$$V = \left[ \frac{1 + \beta}{\beta \alpha - 1} \right] e \sin(\phi),$$

$$\phi = \pm \arccos \left[ \frac{e + \gamma \sqrt{e^2 + \gamma^2 - 1}}{e^2 + \gamma^2} \right],$$

where the auxiliary term $\gamma$ is defined as

$$\gamma = \frac{2e\alpha m\pi}{\alpha + 1} \left[ \frac{1 + \beta}{\beta \alpha - 1} \right].$$

We must emphasize that there are, in principle, four different values for the variable $\phi$. The criterion used to check whether the values are accepted or not, depend on the values of the velocity $V$ since that, by construction, the velocity must be a non-negative number. Thus, we have to choose values of $\phi$ that yields $\sin(\phi) < 0$. Taking into account these considerations and after evaluating the eigenvalues of the Jacobian matrix, say $\lambda_1$ and $\lambda_2$, we obtain that the coordinates of the saddle fixed points are given by Eq. (4) and

$$\phi = -\arccos \left[ \frac{e - \gamma \sqrt{e^2 + \gamma^2 - 1}}{e^2 + \gamma^2} \right].$$

3. Scaling properties

We begin discussing a scaling for the saddle velocity as function of $m$ and $\varepsilon$. It is shown in Fig. 1 the behavior of the saddle velocities $V$ as function of $\varepsilon$ for different values of $m$. We have fixed the values $\beta = 1$ and $\alpha = 0.89$ in the construction of Fig. 1, although similar results would indeed be obtained for other combinations of these parameters too. We can see that all velocity curves behave quite similarly as a function of $\varepsilon$. After the birth, which happens when the velocity $V$ becomes a real number, the saddle velocity stays almost constant for a relative extent range of $\varepsilon$ until approaches to the null value as the control parameter $\varepsilon \to 1$. Based on the behavior shown in Fig. 1, we can suppose that the saddle velocity, at the instant of its
birth, might be described as

\[ V_B(m, e) \propto m^s, \tag{6} \]

where \( s \) is a critical exponent and \( m \) is a positive integer number. The index \( B \) denotes that the velocity is accounted at the birth of the fixed point, i.e. when the control parameters yield both phase \( \phi \) and velocity \( V \) being real numbers. Assuming \( m \) as fixed, the second scaling hypothesis is

\[ V_B(e) \propto e^\nu, \tag{7} \]

where \( \nu \) is also a critical exponent. It is easy to see from Eqs. (6) and (7) that

\[ m \propto e^{\nu/\sigma}. \tag{8} \]

Considering such initial hypotheses we can now formally describe the saddle velocities in terms of a scaling function of the type

\[ V(\epsilon, m) = lV(l^a m, l^b \epsilon), \tag{9} \]

where \( l \) is a scaling factor, \( a \) and \( b \) are scaling exponents. Moreover, the exponents \( a \) and \( b \) must be related to the critical exponents \( \nu \) and \( \sigma \). Since \( l \) is a scaling factor, we can choose it as \( l^a m = 1 \), where we find...
that $l = m^{-1/a}$, and thus Eq. (9) is rewritten as

$$V(m, e) = m^{-1/a} V_1(m^{-b/a} e),$$

(10)

where $V_1(m^{-b/a} e) = V(1, m^{-b/a} e)$. Comparing Eqs. (6) and (10), we obtain that $\sigma = -1/a$.

If we now choose $\theta = 1$, we obtain that $l = e^{-1/b}$. Therefore, Eq. (9) is rewritten as

$$V(m, e) = e^{-1/b} V_2(e^{-a/b} m),$$

(11)

where the auxiliary function $V_2$ is defined as $V_2(e^{-a/b} m) = V(e^{-a/b} m, 1)$. Again, a comparison of Eqs. (7) and (11) gives us that $\nu = -1/b$.

It is worth stressing that all the scaling exponents are determined if we obtain numerically the exponents $\nu$ and $\sigma$. Thus, as an attempt to find such exponents, it is shown in Fig. 2, the behavior of the velocities at the birth of the saddle point both as function of (a) $e$ and (b) $m$. After doing power-law fittings, we found numerically that the critical exponents are $\nu = 1.00154(8) \cong 1$ and $\sigma = -0.99985(2) \cong -1$.

Given the scaling exponents $\nu$ and $\sigma$, a final check of the validity of the scaling hypotheses is the collapse of all curves onto a single and universal plot. Before doing the collapse, let us briefly discuss the procedure used and a property of the function (9). Using Eqs. (8) and (10), it is easy to show that $V(m, e) \propto m^\nu$, thus

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**Fig. 2.** Behavior of the saddle velocities, at their birth, as function of: (a) $e$ and (b) $m$.

**Fig. 3.** (Color online) (a) Behavior of the saddle velocities, at the birth, as function of $e$ for different values of $m$. (b) Their collapse onto a single universal curve.
confirming that only the vertical scale must be changed. With this property in mind, we now made the collapse, as it is shown in Fig. 3. Except on the very near neighboring of the birth of the saddle fixed point, all the curves are perfectly collapsed onto a single and universal plot; thus, confirming that the saddle velocity is indeed a scaling invariant.

4. Final remarks

We have studied a dissipative version of the well-known Fermi–Ulam model. In our approach, we have shown that the saddle velocities can be described using scaling arguments thereby confirming that they are scaling invariant.

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